

1. An automobile manufacturer makes a profit of \$1,500 on the sale of a certain model. It is estimated that for every \$100 of rebate, sales increase by 15%.

(a) What amount of rebate will maximize profit? Use the five-step method, and model as a one-variable optimization problem.

**Step 1: Ask the question.**

Variables:  $r$  = rebate (\$)  
 $s$  = number of cars sold  
 $P$  = profit (\$)

Assumptions:  $s = s_0 (1+0.15(r/100))$   
 $P = (1500-r) s$   
 $s \geq 0, 0 \leq r \leq 1500$   
where the constant  $s_0$  is the number of sales without any rebate

Objective: Maximize  $P$ .

**Step 2: Select the modeling approach.**

We will model this problem as a one variable optimization problem. See text p. 6.

**Step 3: Formulate the model.**

Let  $x=r$  and  $y=P$ , and write

$$y = f(x) = (1500-x) s_0 (1+0.15(x/100)).$$

Our goal is to maximize  $f(x)$  over the interval  $[0, 1,500]$ .

**Step 4: Solve the model.**

Compute  $f'(x) = 1500 s_0 (0.0015) + (-1) s_0 (1+0.15(x/100)) = 0$  at  $x = 416.6667$ ,  
 $f(x) = 1760.42$  and since the graph of  $f(x)$  is a parabola we know this is the global maximum.

**Step 5: Answer the question.**

According to this model, the optimal policy is to offer a rebate of around \$420, which should result in about a 17% increase in profits as compared to no rebate.

(b) Compute the sensitivity of your answer to the 15% assumption. Consider both the amount of rebate and the resulting profit.

Generalize the model from part (a) and let

$$y = f(x) = (1500-x) s_0 (1+e x)$$

where currently  $e = 0.0015$ . Compute that

$f'(x) = 1500 s_0 e + (-1) s_0 (1+e x) = 0$   
at  $x = 750 - 1/(2e)$ . Then  $dx/de = 1/(2 e^2)$ , and so

$S(x,e) = (dx/de) (e/x) = 1/(2*0.0015^2) (0.0015/416.6667) = 0.8$   
so if the rebate is 10% more effective than we thought, then the optimal rebate will be 8% bigger. A similar computation yields  $S(y,e) = 0.38$  so that if the rebate is 10% more effective than we thought, then our optimal profit will be 4% greater.

*(c) Suppose that rebates actually generate only a 10% increase in sales per \$100. What is the effect? What if the response is somewhere between 10 and 15% per \$100 of rebate?*

Using the results of part (b), if  $e$  decreases by 33% to 0.0010 then we expect the optimal rebate to decrease by  $(0.8) 33\% = 27\%$  to around \$310. We would also expect profits to go down by  $(0.38) 33\% = 13\%$  to around \$1530. A direct computation (solve the entire problem again using  $e=0.0010$ ) yields a similar result: a 40% decrease in the optimal rebate (to \$250) and an 11% decrease in profit (to 1562.50).

*(d) Under what circumstances would a rebate offer cause a reduction in profit?*

Using the sensitivity results of part (b), if every \$100 of rebate results in a sales increase of less than 8.3% then the optimal policy is to offer no rebate. To see this, note that currently the optimal profit is 17% higher than with no rebate, and  $(0.38) 44.7\% = 17\%$ . The current rebate effectiveness is  $e = 0.0015$  and so a 44.7% decrease yields 0.00083. Exact calculations yield a similar result: if every \$100 of rebate results in a sales increase of less than 6.7% then the optimal policy is to offer no rebate. To see this, note that by the formula derived in part (b), the optimal rebate is  $x = 750 - 1/(2e)$ , which decreases to zero as  $e$  decreases to  $1/(2*750) = 0.00067$ .

*2. In the pig problem, perform a sensitivity analysis based on the cost per day of keeping the pig. Consider both the effect on the best time to sell and on the resulting profit. If a new feed costing 60 cents/day would let the pig grow at a rate of 7 lbs/day, would it be worth switching feed? What is the minimum improvement in growth rate that would make this new feed worthwhile?*

In this case we have

$y = f(x) = (0.65 - 0.01 x) (200 + 7 x) - 0.60 x$   
and so  $f'(x) = (195 - 14 x) / 100 = 0$  at around  $x = 13.9$ ,  $f(x) = 143.58$ . This is the global maximum since  $f(x)$  is a parabola. Since this is considerably more than the old maximum of 133.20, it is worth while to switch to the new feed.

More generally we have

$y = f(x) = (0.65 - 0.01 x) (200 + g x) - 0.60 x$   
where currently  $g = 7$ . Compute that  $f'(x) = (65 (g - 4) - 2 g x) / 100 = 0$  when  
 $x = 65 (g - 4) / (2 g)$

and substitute this back into the expression for  $y$  to obtain

$$y = 13 (13g^2 + 56g + 208) / (16g)$$

which is equal to 133.20 when  $g = 5.26$ . Thus the new feed would be worth 60 cents per day as long as it produced a growth rate of at least 5.26 pounds per day.

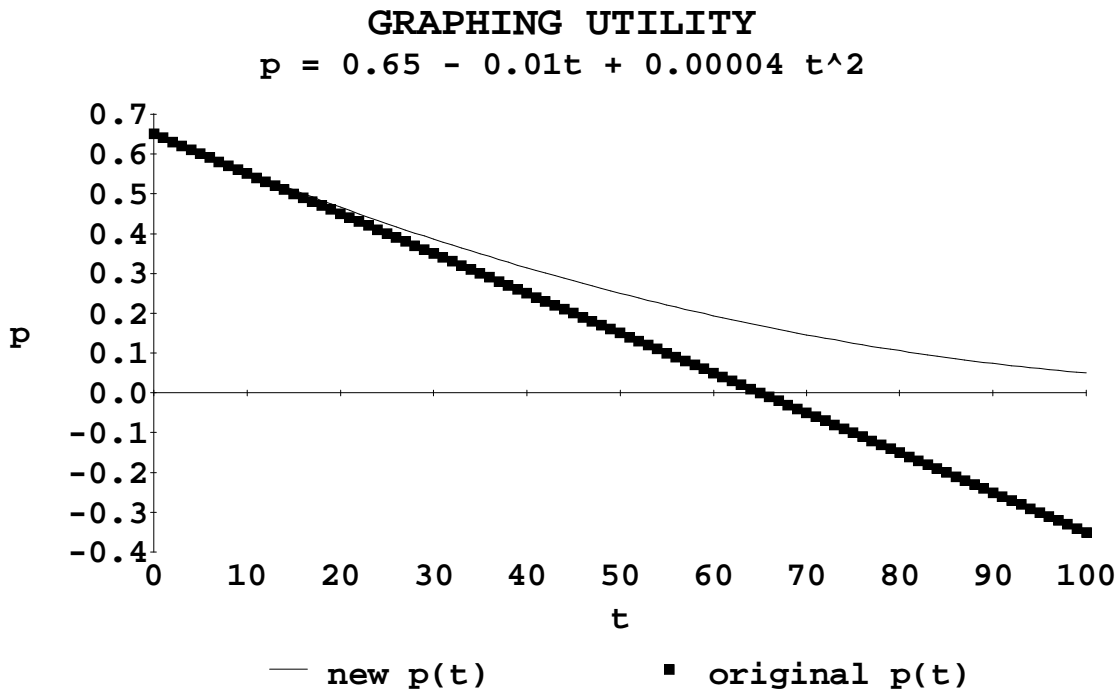
3. Reconsider the pig problem, of Example 1.1, but now assume that the price for pigs is starting to level off. Let

$$p = 0.65 - 0.01t + 0.00004 t^2 \tag{4}$$

represent the price for pigs (cents/lb) after  $t$  days.

(a) Graph Eq. (4) along with our original price equation. Explain why our original price equation could be considered as an approximation to Eq. (4) for values of  $t$  near zero.

The following graph shows the new, nonlinear price function along with the original, linear price function. The original equation is the tangent line to the new equation at  $t = 0$ , and so the original equation may be considered as an approximation of the new equation for values of  $t$  near zero.



(b) Find the best time to sell the pig. Use the five-step method, and model as a one-variable optimization problem.

**Step 1: Ask the question.**

Exactly the same as in the text p. 5, but now we assume  $p = 0.65 - 0.01 t + 0.00004 t^2$ .

**Step 2: Select the modeling approach.**

We will model this problem as a one variable optimization problem. See text p. 6.

**Step 3: Formulate the model.**

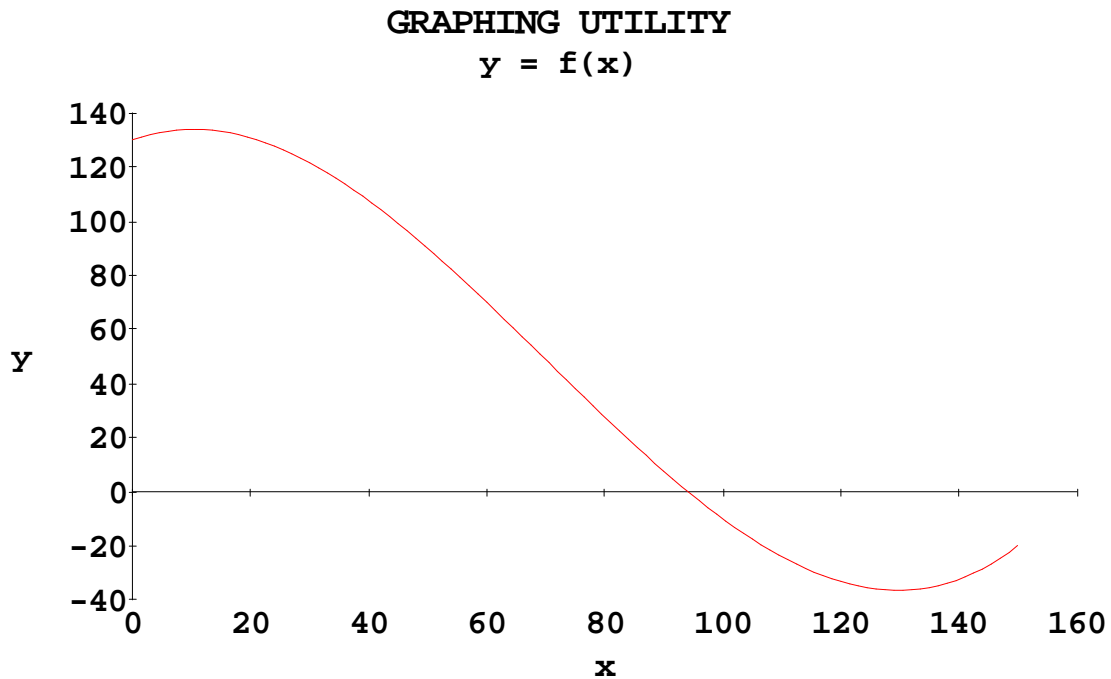
Let  $x=t$  and  $y=P$ , and write

$$y = f(x) = (0.65 - 0.01 x + 0.00004 x^2) (200 + 5 x) - 0.45 x.$$

Our goal is to maximize  $f(x)$  over the set of all  $x \geq 0$ .

**Step 4: Solve the model.**

Compute that  $f'(x) = (3 x^2 - 420 x + 4000)/5000$  which equals zero at  $x = 70 - 10 \cdot \text{SQRT}(321)/3$  and at  $x = 70 + 10 \cdot \text{SQRT}(321)/3$ , or at around  $x = 10.28$  and  $x = 129.72$ . As shown in the following graph, the maximum is at  $x = 10.28$ . Although  $f(x)$  increases to infinity as  $x$  gets large, this is beyond the range where our model makes sense.



**Step 5: Answer the question.**

According to this model, the optimal policy is to sell the pig after 10 days, for a net profit of about \$134.

*(c) The parameter 0.00004 represents the rate at which price is leveling off. Conduct a sensitivity analysis on this parameter. Consider both the optimal time to sell and the resulting profit.*

Generalize the model from part (b) and let

$y = f(x) = (0.65 - 0.01 x + a x^2) (200 + 5 x) - 0.45 x$   
 where currently  $a = 0.00004$ . Compute that

$$f'(x) = (150*a*x^2+x*(4000*a-1)+8)/10 = 0$$

at

$$x = -(\text{SQRT}(16000000*a^2-12800*a+1)+4000*a-1)/(300*a).$$

Then

$$\frac{dx}{de} = \frac{-(\text{SQRT}(16000000*a^2-12800*a+1) + 6400*a-1)}{(300*a^2*\text{SQRT}(16000000*a^2-12800*a+1))}$$

and so  $S(x,e) = (dx/de) (e/x) = 0.31$ . If price is leveling off 10% faster than we thought, then we should wait 3.1% longer to sell the pig. A similar computation yields  $S(y,e) = 0.008$  so that if price is leveling off 1000% faster than we thought, then we should wait 8% longer to sell the pig.

*(d) Compare the results of part (b) to the optimal solution contained in the text. Comment on the robustness of our assumptions about price.*

There is not much difference between the results in part (b) and the results in the text. Our model is reasonably robust with respect to the assumption that price is a linear function of time. Given this, the added computational difficulty associated with the quadratic price model is probably not justified.

*4. An oil spill has fouled 200 miles of Pacific shoreline. The oil company responsible has been given 14 days to clean up the shoreline, after which a fine will be levied in the amount of \$10,000/day. The local clean-up crew can scrub 5 miles of beach per week at a cost of \$500/day. Additional crews can be brought in at a cost of \$18,000 plus \$800/day for each crew.*

*(a) How many additional crews should be brought in to minimize the total cost to the company? Use the five-step method. How much will the clean-up cost?*

**Step 1: Ask the question.**

Variables:  $c$  = number of additional crews  
 $t$  = time to clean up spill (days)  
 $T$  = total cost of clean-up (\$)  
 $F$  = fine (\$)

Assumptions:  $T = 500 t + (18000 + 800 t) c + F$   
 $200 = (5 / 7) (c + 1) t$   
 $F = 0$  if  $t \leq 14$   
 $F = 10,000 (t - 14)$  if  $t > 14$   
 $c$  is a nonnegative integer, and  $t \geq 0$

Objective: Minimize  $T$ .

**Step 2: Select the modeling approach.**

We will model this problem as a one variable optimization problem. See text p. 6.

**Step 3: Formulate the model.**

Let  $x = c$  and  $y = T$ , and write

$$y = f(x) = 500 (280 / (x+1)) + (18,000 + 800 (280 / (x+1))) x$$

if  $x \geq 19$  or

$$y = f(x) = 500 (280 / (x+1)) + (18,000 + 800 (280 / (x+1))) x + 10,000 (280 / (x + 1) - 14)$$

if  $x < 19$ . Our goal is to maximize  $f(x)$  over the set of nonnegative integers  $x$ .

**Step 4: Solve the model.**

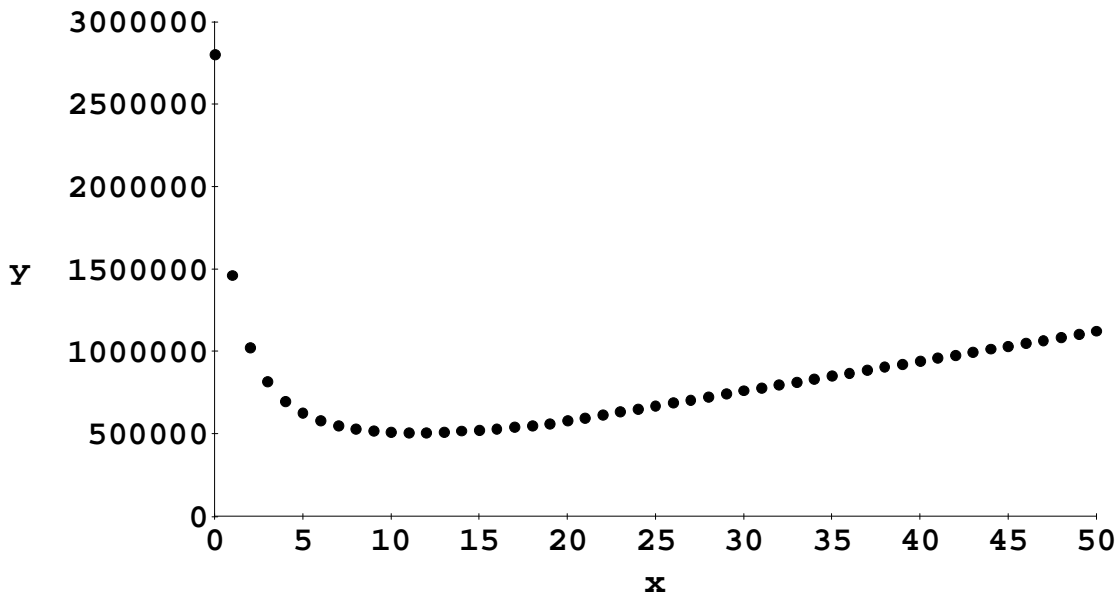
One way to solve is to minimize over the nonnegative reals, and then specialize to the integers. On  $0 \leq x < 19$  we have

$f'(x) = 2000(9x^2 + 18x - 1349)/(x+1)^2$   
 which is negative on  $[0, 11.28)$  and positive on  $(11.28, 19)$ , so  $x = 11.28$  is the minimum. Then the minimum over the integers occurs at either  $x = 11$  or at  $x = 12$ , and we can check that  $x = 11$ ,  $f(x) = 508333$  is smaller. On  $x \geq 19$  we have

$f'(x) = 6000(3x^2 + 6x + 17)/(x+1)^2$   
 which is always positive, and so on this interval the minimum occurs at  $x = 19$ ,  $f(x) = 561800$ . Then the global minimum occurs at  $x = 11$ .

**GRAPHING UTILITY**

$$y = f(x)$$



**Step 5: Answer the question.**

According to this model, the optimal policy is to bring in 11 additional crews, resulting in a total clean-up cost of around \$510,000. Clean-up will take around 23.3 days and the resulting fine will be around \$93,000.

*(b) Examine the sensitivity to the rate at which a crew can clean up the shoreline. Consider both the optimal number of crews and the total cost to the company.*

Generalize the model in part (a) to obtain the assumption

$$200 = r(c + 1)t$$

where currently  $r = (5/7)$  miles per day per crew. Then we have

$$y = f(x) = 500(200 / (r(x+1))) + (18,000 + 800 * 200 / (r(x+1)))x$$

if  $x \geq 19$  or

$$y = f(x) = 500(200 / (r(x+1))) + (18,000 + 800 * 200 / (r(x+1)))x + 10,000(200 / (r(x+1)) - 14)$$

if  $x < 19$ . For values of  $r$  near  $(5/7)$  the minimum should still occur on the interval  $(0, 19)$ . Solving

$$0 = f'(x) = 2000 * (9 * r * x^2 + 18 * r * x + 9 * r - 970) / (r * (x+1)^2)$$

yields

$$x = (\text{SQRT}(970) - 3 * \text{SQRT}(r)) / (3 * \text{SQRT}(r))$$

and so

$$dx / dr = -\text{SQRT}(970) / (6 * r^{3/2}).$$

Substituting  $r = (5/7)$  we obtain

$$S(x, r) = (dx / dr) (r / x) = -0.54$$

so that if the cleanup crews are 10% faster than expected, the optimal number of crews decreases by about 5.4%. If we substitute the formula for the optimal  $x$  in terms of  $r$  into the formula for  $y = f(x)$  in the case  $x < 19$ , we obtain

$$y = -2000 * (79 * r - 6 * \text{SQRT}(970) * \text{SQRT}(r) - 80) / r$$

and then we can also calculate

$$dy / dr = -2000 * \text{SQRT}(10) * (3 * \text{SQRT}(97) * \text{SQRT}(r) + 8 * \text{SQRT}(10)) / r^2$$

Substituting  $r = (5/7)$  we obtain

$$S(y, r) = (dy / dr) (r / y) = -0.88$$

so that if the cleanup crews are 10% faster than expected, the total cost of clean-up decreases by about 8.8%.

*(c) Examine the sensitivity to the amount of the fine. Consider the number of days the company will take to clean up the spill and the total cost to the company.*

Generalize the model in part (a) to obtain the assumption

$$F = a(t - 14) \quad \text{if } t > 14$$

where currently  $a = 10,000$  dollars per day. Then for values of  $a$  near 10,000 we have

$$y = f(x) = 500(280 / (x+1)) + (18,000 + 800(280 / (x+1)))x + a(280 / (x+1) - 14)$$

if  $x < 19$ . The optimal number of crews is  $x = \text{SQRT}(14) * \text{SQRT}(a-300) / 30 - 1$  and so the time to finish the clean-up is

$$t = 280 / (x+1) = 600 * \text{SQRT}(14) / \text{SQRT}(a-300)$$

and then we can compute that at  $a=10,000$  we have

$$S(t,a) = (dt/da)(a/t) = (-300 * \text{SQRT}(14) / (a-300)^{(3/2)}) * (a/t) = -0.52$$

so that if the fine is raised by 2% then the cleanup time should decrease by about 1%.

Substituting the optimal formula for  $x$  in terms of  $a$  into the equation for  $y$  above, we obtain

$$y = 2 * (600 * \text{SQRT}(14) * \text{SQRT}(a-300) - 7 * a + 103000)$$

$$dy/da = 2 * \text{SQRT}(7) * (300 * \text{SQRT}(2) - \text{SQRT}(7) * \text{SQRT}(a-300)) / \text{SQRT}(a-300)$$

so that at  $a = 10,000$  we have

$$S(y,a) = (dy/da)(a/y) = 0.17$$

which means that if the fine is increased then the total cleanup cost to the company will go up by about 1.7% for each additional \$1,000 per day of fine.

*(d) The company has filed an appeal on the grounds that the amount of the fine is excessive. Assuming that the only purpose of the fine is to motivate the company to clean up the oil spill in a timely manner, is the fine excessive?*

Reasonable answers will differ on this question. On the one hand, the fine is only 19% of the total cost, and if the fine were reduced by 50% then the number of days to clean up the spill would increase by about 25% and it would only save the company about 8.5% of the total cost. So the amount of the fine does not seem excessive. On the other hand, if the 14 day limit were extended to 21 days, cleanup would proceed exactly as before, only the company would save \$70,000. So in this case the 14 day limit does seem excessive.

*5. It is estimated that the growth rate of the fin whale population (per year) is  $r x (1 - x / K)$ , where  $r = 0.08$  is the intrinsic growth rate,  $K = 400,000$  is the maximum sustainable population, and  $x$  is the current population, now around 70,000. It is further estimated that the number of whales harvested per year is about  $.00001 E x$ , where  $E$  is the level of fishing effort in boat-days. Given a fixed level of effort, population will eventually stabilize at the level where growth rate equals harvest rate.*

*(a) What level of effort will maximize the sustained harvest rate? Model as a one-variable optimization problem using the five-step method.*

### **Step 1: Ask the question.**

Variables:  $x$  = population (whales)  
 $E$  = level of effort (boat-days)  
 $g$  = growth rate (whales per year)  
 $h$  = harvest rate (whales per year)

Assumptions:  $g = 0.08 x (1 - x / 400,000)$   
 $h = .00001 E x$   
 $g = h, \quad x \geq 0, E \geq 0$

Objective: Maximize  $h$ .

**Step 2: Select the modeling approach.**

We will model this problem as a one variable optimization problem. See text p. 6.

**Step 3: Formulate the model.**

Let  $y = h$ , and write

$$y = f(x) = 0.08 x (1 - x / 400,000).$$

Our goal is to maximize  $f(x)$  over the interval  $x \geq 0$ .

**Step 4: Solve the model.**

Compute  $f'(x) = 0$  at  $x = 200,000$ ,  $f(x) = 8000$  and since the graph of  $f(x)$  is a parabola we know this is the global maximum.

**Step 5: Answer the question.**

According to this model, the optimal policy is to harvest 8000 whales per year, which requires controlling the level of effort at 4000 boat-days per year. This will maintain the population of whales at 200,000 which is higher than the current population. This indicates that in the past the harvesting rate has exceeded the optimum level according to this model.

*(b) Examine the sensitivity to the intrinsic growth rate. Consider both the optimum level of effort and the resulting population level.*

Generalize the model in part (a) to obtain the assumption

$$g = r x (1 - x / 400,000)$$

where currently  $r = 0.08$ . Then we have

$$y = f(x) = r x (1 - x / 400,000)$$

and the optimum is still at  $x = 200,000$  but now  $f(x) = 100,000 r$  which leads to

$E = 50,000 r$ . Then

$$S(x, r) = 0$$

$$S(E, r) = 1$$

because  $x$  does not depend on  $r$ , and  $E$  is proportional to  $r$ .

*(c) Examine the sensitivity to the maximum sustainable population. Consider both the optimum level of effort and the resulting population level.*

Generalize the model in part (a) to obtain the assumption

$$g = 0.08 x (1 - x / K)$$

where currently  $K = 400,000$ . Then we have

$$y = f(x) = 0.08 x (1 - x / K)$$

and the optimum is at  $x = K / 2$ ,  $f(x) = 0.02 K$  which leads to  $E = 4000$ . Then

$$S(x, K) = 1$$

$$S(E, K) = 0$$

because  $E$  does not depend on  $K$ , and  $x$  is proportional to  $K$ .

6. In problem 5, suppose that the cost of whaling is \$500 per boat-day, and the price of a fin whale carcass is \$6,000.

(a) Find the level of effort that will maximize profit over the long term. Model as a one-variable optimization problem using the five-step method.

### Step 1: Ask the question.

Variables:

- $x$  = population (whales)
- $E$  = level of effort (boat-days)
- $g$  = growth rate (whales per year)
- $h$  = harvest rate (whales per year)
- $R$  = revenue (dollars per year)
- $C$  = cost (dollars per year)
- $P$  = profit (dollars per year)

Assumptions:

- $g = 0.08 x (1 - x / 400,000)$
- $h = .00001 E x$
- $R = 6000 h$
- $C = 500 E$
- $P = R - C$
- $g = h, \quad x \geq 0, E \geq 0$

Objective: Maximize  $P$ .

### Step 2: Select the modeling approach.

We will model this problem as a one variable optimization problem. See text p. 6.

### Step 3: Formulate the model.

Setting  $g = h$  we see that

$$\begin{aligned} E &= 0.08 (1 - x / 400,000) / .00001 \\ &= 8000 - .02 x \end{aligned}$$

Let  $y = P$ , and write

$$\begin{aligned} y = f(x) &= 6000 (.00001 E x) - 500 E \\ &= (.06 x - 500) E \\ &= -.0012 x^2 + 490 x - 4,000,000. \end{aligned}$$

Our goal is to maximize  $f(x)$  over the interval  $x \geq 0$ .

#### Step 4: Solve the model.

Compute  $f'(x) = 0$  at  $x = 204,167$ ,  $f(x) = 4.60208 \cdot 10^7$  and since the graph of  $f(x)$  is a parabola we know this is the global maximum.

#### Step 5: Answer the question.

According to this model, the optimal policy is to harvest 7997 whales per year, which requires controlling the level of effort at 3917 boat-days per year. This will maintain the population of whales at 204,167 and will net the industry an annual profit of around 46 million dollars.

*(b) Examine the sensitivity to the cost of whaling. Consider both the eventual profit in \$ / year and the level of effort.*

Generalize the model in part (a) to obtain the assumption

$$\begin{aligned}y &= f(x) = 6000 (.00001 E x) - w E \\ &= (.06 x - w) E \\ &= -.0012 x^2 + (480 + .02 w) x - 8000 w.\end{aligned}$$

where currently  $w = 500$ . Then the optimum is at  $x = 200,000 + 25w/3$ ,

$P = f(x) = (1/12) (w - 24000)^2$  which leads to  $E = (24000 - w)/6$ . Then

$$S(P, w) = (dP/dw) (w/P) = -0.04$$

$$S(E, w) = (dE/dw) (w/E) = -0.02$$

so if the cost of whaling goes up by 10% then the optimal profit decreases by 0.4% and the optimal level of effort decreases by 0.2 %.

*(c) Examine the sensitivity to the price of a fin whale carcass. Consider both profit and level of effort.*

Generalize the model in part (a) to obtain the assumption

$$\begin{aligned}y &= f(x) = c (.00001 E x) - 500 E \\ &= (.00001 c x - 500) E \\ &= -(c/5,000,000) x^2 + (2c/25 + 10) x - 4,000,000.\end{aligned}$$

where currently  $c = 6000$ . Then the optimum is at  $x = 200,000 (c + 125)/c$ ,

$f(x) = 8000 (c^2 - 250 c + 15625) / c$  which leads to  $E = 4000(c-125)/c$ . Then

$$S(P, c) = (dy/dc) (c/y) = 1.04$$

$$S(E, c) = (dE/dc) (c/E) = 0.02$$

so if the price of a fin whale carcass goes up by 10% then the optimal profit increases by about 10.4% and the optimal level of effort increases by 0.2 % .

*(d) Over the past 30 years there have been several unsuccessful attempts to ban whaling worldwide. Examine the economic incentives for whalers to continue harvesting. In particular, determine the conditions (values of the two parameters: cost per boat-day and price per fin whale carcass) under which harvesting the fin whale produces a sustained profit over the long term.*

From part (b) we see that the optimal profit is  $P = f(x) = (1/12) (w - 24000)^2$  at \$6000 per carcass, so that the industry makes a profit whenever the cost of whaling is below \$24,000 per boat-day. From part (c) we see that  $P = 8000 (c^2 - 250c + 15625) / c$  at \$500 per boat-day, in which case the industry makes a profit whenever the price per carcass exceeds \$125. It is difficult analytically to consider the optimal P as a function of both c and w together, but using our sensitivity results we have approximately that

$P = 46,000,000 + 1.04 (46,000/6) (c-6,000) - 0.04 (460,000/5) (w-500)$   
 and then  $P > 0$  whenever  $c > 6w/13$  or in other words the industry makes a profit as long as the price of a fin whale carcass is a bit more than half of the cost per boat-day of whaling. Thus there is a very strong profit motive to continue whaling.

7. Reconsider the pig problem of Example 1.1, but now suppose that our objective is to maximize our profit rate (\$/day). Assume that we have already owned the pig for 90 days and have invested \$100 in this pig to date.

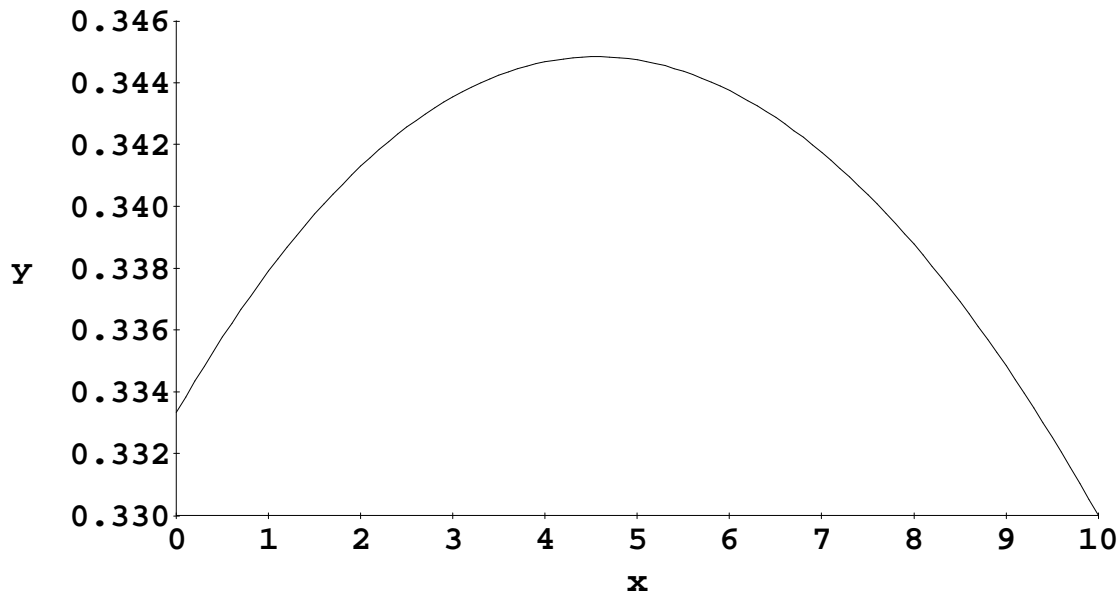
(a) Find the best time to sell the pig. Use the five-step method, and model as a one-variable optimization problem.

Step one is the same as in figure 1.1 of the text, except that we add a new variable Q = profit per day (\$/day), we assume  $C = 100 + 0.45t$ ,  $Q = P / (t + 90)$ , and our objective is to optimize profit per day. This is a one variable optimization problem. Letting  $x = t$  and  $y = f(x) = Q$  we are to find the maximum of the function

$f(x) = ((200 + 5x)(0.65 - 0.01x) - (100 + 0.45x)) / (x + 90)$   
 over the set of all nonnegative x. The graph indicates a maximum around  $x = 4.5$ ,  $f(x) = 0.345$ .

### GRAPHING UTILITY

$$y = f(x)$$



Compute

$f'(x) = -(x^2 + 180x - 840) / (20(x+90)^2) = 0$   
 at  $x = 2\sqrt{2235} - 90$  or approximately  $x = 4.55$ . Then the farmer should sell the pig in 4 or 5 days to maximize profit per day, or the rate at which the farmer earns income.

(b) Examine the sensitivity to the growth rate of the pig. Consider both the best time to sell and the resulting profit rate.

Let  $g$  denote the growth rate of the pig, where currently we assume  $g = 5$  lbs / day. Now we are to find the maximum of the function

$$f(x) = ((200 + gx)(0.65 - 0.01x) - (100 + 0.45x)) / (x + 90)$$

over the set of all nonnegative  $x$ . Compute

$$f'(x) = -(g^2x^2 + 180gx - 150(39g - 167)) / (100(x+90)^2) = 0$$

at

$$x = 5\sqrt{6}(\sqrt{93g - 167} - 3\sqrt{6}\sqrt{g}) / \sqrt{g}$$

and then at  $g = 5$  we have

$S(x, g) = (dx / dg)(g / x) = 4.82$   
 so that if the pig grows 1% faster than expected, we should wait 5% longer to sell the pig. Substituting into  $y = f(x)$  we can also compute that

$S(y, g) = (dy / dg)(g / y) = 0.42$   
 so that if the pig grows 10% faster than expected we should gain an additional 4% profit per day.

(c) Examine the sensitivity to the rate at which the price for pigs is dropping. Consider both the best time to sell and the resulting profit rate.

Let  $r$  denote the rate at which price is falling, where currently  $r = 0.01$  (\$/day). Then we are to maximize

$$f(x) = ((200 + 5x)(0.65 - rx) - (100 + 0.45x)) / (x + 90)$$

over the set of all nonnegative  $x$ . Compute

$$f'(x) = -(5r^2x^2 + 900rx + 6(3000r - 37)) / (x+90)^2 = 0$$

at

$$x = \sqrt{30}(\sqrt{3750r + 37} - 15\sqrt{30}\sqrt{r}) / (5\sqrt{r})$$

and then at  $r = 0.01$  we have

$S(x, r) = (dx / dr)(r / x) = -5.16$   
 so that if the price drops 1% faster than expected, we should sell the pig 5% sooner. Substituting into  $y = f(x)$  we can also compute that

$S(y, r) = (dy / dr)(r / y) = -0.31$   
 so that if the price drops 10% faster than expected then we will lose about 3% of our expected profit per day.

8. Reconsider the pig problem of Example 1.1, but now take into account the fact that the growth rate of the pig decreases as the pig gets older. Assume that the pig will be fully grown in another five months.

(a) Find the best time to sell the pig in order to maximize profit. Use the five-step method, and model as a one-variable optimization problem.

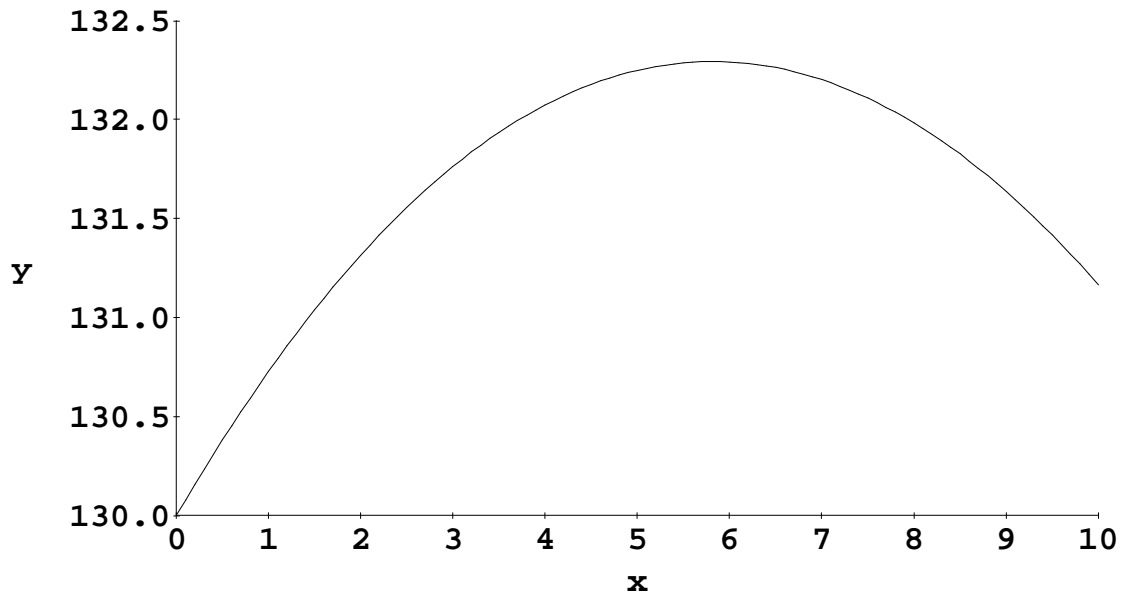
The results of step one are the same as in figure 1.1 of the text, except that now we assume the growth rate of the pig is  $r$  (lbs/day) where  $r = 5 - t / 30$  so that the weight  $w$  (lbs) of the pig after  $t$  days is  $w = 200 + (5 - t / 30) t$ . Then we need to maximize

$$f(x) = (200 + (5 - x / 30) x) (0.65 - 0.01 x) - 0.45 x$$

over the set of all nonnegative  $x$ . The graph indicates a maximum around  $x = 6$ ,  $f(x) = 132$ .

### GRAPHING UTILITY

$$y = f(x)$$



We compute that

$$f'(x) = (3x^2 - 430x + 2400) / 3000 = 0$$

at  $x = 215/3 - 5\sqrt{1561}/3 = 5.82$  so that  $f(x) = 132.29$ . Then the farmer should sell the pig after 6 days, and the expected net profit will be about \$132.

(b) Examine the sensitivity to the time it will take until the pig is fully grown. Consider both the best time to sell and the resulting profit.

We generalize our previous model. Let  $a$  denote the rate at which the growth of the pig slows. Currently  $a = 1.0$  lbs/day per month. Now the problem is to maximize

$$f(x) = (200 + (5 - ax / 30) x) (0.65 - 0.01 x) - 0.45 x$$

over the set of all nonnegative  $x$ . For values of  $a$  near 1.0 the maximum should occur at a point near  $x = 6$  where  $f'(x) = 0$ . We compute that

$$f'(x) = (3ax^2 - 10x(13a + 30) + 2400) / 3000 = 0$$

at  $x = -5(\sqrt{169a^2 + 492a + 900} - 13a - 30) / (3a)$ . Then at  $a = 1.0$  we have

$$S(x, a) = (dx / da) (a / x) = -0.28$$

so that if the pig stops growing 10% sooner than expected, then we should sell the pig 3% sooner. Substituting into the formula for  $y = f(x)$  we may also compute that

$$S(y, a) = (dy / da) (a / y) = -0.005$$

so that the resulting profit is almost totally insensitive to the rate at which the pig stops growing. This makes sense because the growth rate of the pig will not change much in the 6 or so days until we sell.

9. A local daily newspaper with a circulation of 80,000 subscribers is thinking of raising its subscription price. Currently the price is \$1.50 per week, and it is estimated that the paper would lose 5,000 subscribers if the rate were to be raised by 10 cents/week.

(a) Find the subscription price that maximizes profit. Use the five-step method, and model as a one-variable optimization problem.

### Step 1: Ask the question.

Variables:     $p$         = subscription price (\$/paper)  
                   $s$         = number of subscriptions (papers)  
                   $P$         = profit (\$)

Assumptions:  $s$         =  $80000 - 50000(p - 1.50)$   
                   $P$         =  $p s$   
                   $s$          $\geq 0$   
                   $p$          $\geq 0$

Objective:    Maximize  $P$ .

### Step 2: Select the modeling approach.

We will model this problem as a one variable optimization problem. See text p. 6.

### Step 3: Formulate the model.

Let  $x = p$  and  $y = P$ , and write

$$y = f(x) = x(80000 - 50000(x - 1.50)).$$

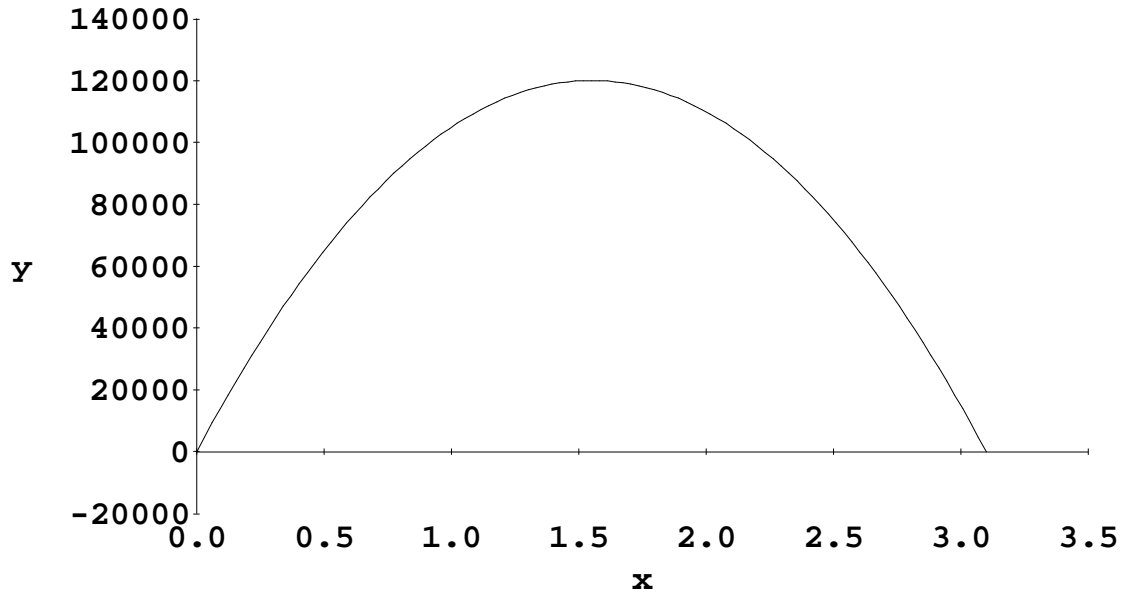
Our goal is to maximize  $f(x)$  over the interval  $[0, 3.10]$  since these are the only values of  $x$  that satisfy both  $s \geq 0$  and  $p \geq 0$ . In other words, price cannot be negative, and according to our model the number of subscriptions drops to zero when price is raised to \$3.10.

### Step 4: Solve the model.

A graph of the function  $f(x)$  shows that the maximum occurs at around  $x = 1.5$  and  $f(x) = 120,000$ .

## GRAPHING UTILITY

$$y = f(x)$$



Compute  $f'(x) = 5000*(31-20*x) = 0$  at  $x = 31/20 = 1.55$ ,  $f(x) = 120125$  which is the global maximum.

### Step 5: Answer the question.

According to this model, the optimal policy is to raise the price of the paper by 5 cents to \$1.55 per week. The resulting profit (actually it is revenue since we have not accounted for any costs in our model) is \$120,125 per week as opposed to \$120,000 currently.

*(b) Examine the sensitivity of your answer in part (a) to the assumption of 5,000 lost subscribers. Calculate the optimal subscription rate assuming that this parameter is 3,000, 4,000, 5,000, 6,000, or 7,000.*

We repeat the above procedure with the parameter  $10n = 50000$  replaced by 30000, ... to obtain the following results:

n	x	y
3000	2.08	130210
4000	1.75	122490
5000	1.55	120125
6000	1.42	120410
7000	1.32	122220

(c) Let  $n = 5,000$  denote the number of subscribers lost when the subscription price increases by 10 cents. Calculate the optimal subscription price  $p$  as a function of  $n$ , and use this formula to determine the sensitivity  $S(p,n)$ .

Now we need to maximize

$$f(x) = x(80000 - 10n(x - 1.50))$$

where currently  $n = 5000$ . For values of  $n$  near 5000 the maximum should occur at the point near  $x = 1.50$  where  $f'(x) = 0$ . We calculate that

$$f'(x) = -5*(4*n*x - 3*n - 16000) = 0$$

at  $p = x = (3*n + 16000)/(4*n)$ . Then at  $n = 5000$  we find that

$$S(p, n) = (dp / dn) (n / p) = -16000/(3*n + 16000) = -16/31 = -0.52$$

so that if the number  $n$  of lost subscriptions for a 10 cent price increase is 20% higher than expected, then the optimal price is about 10% lower. This is in rough agreement with the results of part (b) above.

(d) *Should the paper change its subscription price? Justify your conclusions in plain English.*

The newspaper should not make a change in its subscription price based on the results of this model. The chart in part (b) shows that for current estimate of  $n = 5000$  we are already very close to the optimal subscription price. In fact the results of part (a) show that we are within 5 cents. If we did raise the price by 5 cents we would only increase our revenue by an estimated \$125 or about 0.10 percent. The likely magnitude of error in our estimate of  $n$  is probably at least 10 or 20 percent, and so in fact the optimal price may be slightly higher or lower than the current price of \$1.50 per week.