Solutions

Section 1-1.

1. $A\hat{v} = \lambda\hat{v}$ so $A(\alpha\hat{v}) = \alpha(A\hat{v}) = \alpha(\lambda\hat{v}) = \lambda(\alpha\hat{v})$.

2. $\langle f, g+h\rangle = \overline{\langle g+h, f\rangle} = \overline{\langle g,f\rangle + \langle h,f\rangle} = \overline{\langle g,f\rangle} + \overline{\langle h,f\rangle} = \langle f,g\rangle + \langle f,h\rangle$.

3. $L[y_1+y_2] = a_0(x)[y_1+y_2] + a_1(x)[y_1+y_2]' + a_2(x)[y_1+y_2]'' =$

$a_0(x)y_1 + a_0(x)y_2 + a_1(x)y_1' + a_1(x)y_2' + a_2(x)y_1'' + a_2(x)y_2'' =$

$a_0(x)y_1 + a_1(x)y_1' + a_2(x)y_1'' + a_0(x)y_2 + a_1(x)y_2' + a_2(x)y_2'' = L[y_1] + L[y_2]$.

$L[\alpha y] = a_0(x)(\alpha y) + a_1(x)(\alpha y)' + a_2(x)(\alpha y)'' =$

$\alpha[a_0(x)(y) + a_1(x)(y)' + a_2(x)(y)''] = \alpha L[y]$.

4. $\langle \alpha f, g\rangle = \int_a^b \alpha f(x)\overline{g(x)}\,dx = \alpha\int_a^b f(x)\overline{g(x)}\,dx = \alpha\langle f,g\rangle$.

$$\langle f, g+h\rangle = \int_a^b f(x)\overline{[g(x)+h(x)]}dx = \int_a^b f(x)\overline{[g(x)]}dx + \int_a^b f(x)\overline{[h(x)]}dx = \langle f,g\rangle + \langle f,h\rangle.$$

$$\overline{\langle f,g\rangle} = \overline{\int_a^b f(x)\overline{g(x)}\,dx}$$

$$= \int_a^b \overline{f(x)\overline{g(x)}}dx = \int_a^b \overline{f(x)}g(x)dx = \int_a^b g(x)\overline{f(x)}\,dx = \langle g,f\rangle.$$

$$\langle f,f\rangle = \int_a^b |f(x)|^2 dx \ge 0.$$

Note that if $f(x)$ is continuous and $f(x)$ is not identically 0 on $[a,b]$, then there is an interval in $[a,b]$ where $|f(x)| > 0$. Thus

$$\int_a^b |f(x)|^2 dx > 0.$$

This is not necessarily true if we do not require $f(x)$ to be continuous. For example

$$f(x) = \begin{cases} 0 \; if \; x \neq a \\ 1 \; \; if \; x = a \end{cases}.$$

In a setting that uses measure theory, one groups into a single class functions that are equal "almost everywhere" and we can then not require the functions to be continuous but merely integrable.

5. This is nearly identical to problem 4. We require $w(x) > 0$ to ensure $\langle f, f \rangle_w > 0$ if $f \neq 0$.

6. (b.)

$$\hat{u} \cdot \hat{v} = \frac{\sqrt{3}}{4} + \frac{\sqrt{15}}{4} + 5\sqrt{24} \approx 25.89; \; \frac{25.89}{26} \approx .995; \; \cos^{-1}.995 \approx .092 \text{ radians.}$$

(c.) The arc length s subtended by an angle θ (measured in radians) of a circle of radius r is $s = r\theta$. So $s = \sqrt{26}\text{feet}(.092) \approx .469$ feet.

7. $$\langle Ly_1, y_2 \rangle = \int_{-1}^{1} [Ly_1(x)]\, y_2(x) dx = \int_{-1}^{1} [(1 - x^2) y_1''(x) - 2x y_1'(x)]\, y_2(x) dx.$$

Integrate $\int_{-1}^{1} [(1 - x^2) y_1''(x)]\, y_2(x) dx$ by parts with

$$u = (1 - x^2) y_2(x), \quad du = y_2'(x) - 2x y_2(x) - x^2 y_2'(x)$$

$$dv = y_1''(x), \quad v = y_1'(x).$$

Then

$$\int_{-1}^{1} [(1 - x^2) y_1''(x)]\, y_2(x) dx$$

$$= (1 - x^2) y_2(x) y_1'(x) |_{-1}^{1} - \int_{-1}^{1} [y_1'(x)\, y_2'(x) - 2x y_1'(x) y_2(x)$$
$$- x^2 y_2'(x) y_1'(x)] dx$$
$$= -\int_{-1}^{1} [y_1'(x)\, y_2'(x) - 2x y_1'(x) y_2(x) - x^2 y_2'(x) y_1'(x)] dx.$$

Thus

$$\int_{-1}^{1}[(1-x^2)y_1''(x)-2xy_1'(x)]\,y_2(x)dx$$

$$=-\int_{-1}^{1}[y_1'(x)\,y_2'(x)-2xy_1'(x)y_2(x)-x^2y_2'(x)y_1'(x)]dx$$

$$-\int_{-1}^{1}2xy_1'(x)y_2(x)dx=-\int_{-1}^{1}(1-x^2)\,y_2'(x)y_1'(x)]dx.$$

Similarly,

$$\langle y_1, Ly_2\rangle=\int_{-1}^{1}y_1(x)[(1-x^2)y_2''(x)-2xy_2'(x)]\,dx=-\int_{-1}^{1}(1-x^2)\,y_1'(x)y_2'(x)]dx.$$

8. For each part $f(x)=A\cos(\sqrt{\lambda}x)+B\sin(\sqrt{\lambda}x)$. We find λ and A and B that satisfy the boundary conditions.

(a.) $f(0)=A$ so $A=0$ and $f(x)=\ B\sin(\sqrt{\lambda}x)$. $f(\pi)=B\sin(\sqrt{\lambda}\pi)=0$ so $\sqrt{\lambda}\pi=n\pi$ and so $\lambda=n^2$. Thus the eigenfunctions are $B\sin n\,x$.

(b.) $f(0)=A$ so $A=0$ and $f(x)=\ B\sin(\sqrt{\lambda}x)$. $f(L)=B\sin(\sqrt{\lambda}L)=0$ so $\sqrt{\lambda}L=n\pi$ and so $\lambda=(n\pi/L)^2$. Thus the eigenfunctions are $B\sin(n\pi/L)\,x$.

(c.) $f(-\pi/2)=A\cos(-\sqrt{\lambda}\pi/2)+B\sin(-\sqrt{\lambda}\pi/2)=A\cos(\sqrt{\lambda}\pi/2)-B\sin(\sqrt{\lambda}\pi/2)=0$

$$f(\pi/2)=A\cos(\sqrt{\lambda}\pi/2)+B\sin(\sqrt{\lambda}\pi/2)=0.$$

$$2\,A\cos\left(\frac{\sqrt{\lambda}\pi}{2}\right)=0\text{ so }A=0\text{ or }\sqrt{\lambda}=2n+1;\ \text{ also }\ 2B\sin\left(\frac{\sqrt{\lambda}\pi}{2}\right)=0\ \text{ so }\ B=0\text{ or }\sqrt{\lambda}=2n.$$

Thus, the eigenfunctions are $f(x)=\cos[(2n+1)x]$ and $f(x)=\sin(2nx)$.

(d.) $f(-L)=A\cos(-\sqrt{\lambda}L)+B\sin(-\sqrt{\lambda}L)=A\cos(\sqrt{\lambda}L)-B\sin(\sqrt{\lambda}L)=0$

$$f(\pi/2)=A\cos(\sqrt{\lambda}L)+B\sin(\sqrt{\lambda}L)=0.$$

$$2\,A\cos(\sqrt{\lambda}L)=0\text{ so }A=0\text{ or }\sqrt{\lambda}=\frac{(n+\frac{1}{2})\pi}{L};\ \text{ also }\ 2B\sin(\sqrt{\lambda}L)=0\ \text{ so }\ B=0\text{ or }\sqrt{\lambda}=\frac{n\pi}{L}.$$

9. We know (by definition of a basis) every vector in V can be written as a linear combination of $\hat{x}_1,\dots,\hat{x}_n$. We show the representation is unique. Suppose

$$\hat{v} = a_1\hat{x}_1 + \cdots + a_n\hat{x}_n \text{ and } \hat{v} = b_1\hat{x}_1 + \cdots + b_n\hat{x}_n.$$

Then $\hat{0} = \hat{v} - \hat{v} = (a_1 - b_1)\hat{x}_1 + \cdots + (a_n - b_n)\hat{x}_n$. Since $\{\hat{x}_1, \ldots, \hat{x}_n\}$ is a linearly independent set, each $a_i - b_i = 0$.

10. If $\{\hat{x}_1, \ldots, \hat{x}_n\}$ is an orthonormal basis then $\sum_{i=1}^{n} a_i\, b_i\langle\hat{x}_i, \hat{x}_i\rangle = \sum_{i=1}^{n} a_i\, b_i$.

11. $(a.)$ If $T(\hat{x}) = T(\hat{y})$ then $\hat{0} = T(\hat{x}) - T(\hat{y}) = (\hat{x} - \hat{y})$ so $\hat{x} = \hat{y}$.

(b.) We show $\{T(\hat{x}_1), \ldots, T(\hat{x}_n)\}$ is a linearly independent set. Suppose

$$\hat{0} = a_1T(\hat{x}_1) + \cdots + a_nT(\hat{x}_n) = T(a_1\hat{x}_1 + \cdots + a_n\hat{x}_n). \text{Then } a_1\hat{x}_1 + \cdots + a_n\hat{x}_n =$$

$\hat{0}$. Since $\{\hat{x}_1, \ldots, \hat{x}_n\}$ is a basis, each $a_i = 0$.
In an $n-$dimensional vector space, any set of n linearly independent vectors is a basis.

12.

$T(f) = af''(t) + bf'(t)$ is linearly independent,

$T(f) = af''(t) + bf'(t) + 1$ is not linearly independent,

$T(f) = e^t f'(t)$ is linearly independent,

$T(f) = (f(t))^2$ is not linearly independent.

13. We have $\langle x_1, x_1\rangle = 1$, $\langle x_2, x_2\rangle = 1$, $\langle x_1, x_2\rangle = 0$, $\langle Tx_1, Tx_1\rangle = 1$, $\langle Tx_2, Tx_2\rangle = 1$, $\langle Tx_1, Tx_2\rangle = 0$.

Let $\hat{x} = ax_1 + bx_2$. Then in the real number case, (for complex numbers we must introduce complex conjugates but the technique is the same)

$$\langle\hat{x}, \hat{x}\rangle = \langle ax_1 + bx_2, ax_1 + bx_2\rangle = a^2\langle x_1, x_1\rangle + 2ab\langle x_1, x_2\rangle + b^2\langle x_2, x_2\rangle = a^2 + b^2$$

$$\langle T\hat{x}, T\hat{x}\rangle = \langle T(ax_1 + bx_2), T(ax_1 + bx_2)\rangle = \langle aT(x_1) + bT(x_2), aT(x_1) + bT(x_2)\rangle$$
$$= a^2\langle Tx_1, Tx_1\rangle + 2ab\langle Tx_1, Tx_2\rangle + b^2\langle Tx_2, Tx_2\rangle = a^2 + b^2.$$

14. If

$$T(\hat{x}_i) = a_{1i}\hat{x}_1 + \cdots + a_{ni}\hat{x}_n = \lambda_i\hat{x}_i \text{ then } a_{ji} = \begin{cases} 0 \; if \; j \neq i \\ \lambda_i \; if \; i = j \end{cases}$$

15. (a.) Since we are in the real numbers $\langle\hat{x}, \hat{y}\rangle = \langle\hat{y}, \hat{x}\rangle$. Then

$$\langle U(\hat{x}+\hat{y}), U(\hat{x}+\hat{y})\rangle = \langle U\hat{x}+U\hat{y}, U\hat{x}+U\hat{y}\rangle = \langle U\hat{x}, U\hat{x}\rangle + 2\langle U\hat{x}, U\hat{y}\rangle + \langle U\hat{y}, U\hat{y}\rangle$$
$$= \langle \hat{x}, \hat{x}\rangle + 2\langle U\hat{x}, U\hat{y}\rangle + \langle \hat{y}, \hat{y}\rangle$$

$$\langle (\hat{x}+\hat{y}), (\hat{x}+\hat{y})\rangle = \langle \hat{x}+\hat{y}, \hat{x}+\hat{y}\rangle = \langle \hat{x}, \hat{x}\rangle + 2\langle \hat{x}, \hat{y}\rangle + \langle \hat{y}, \hat{y}\rangle$$

and since $\langle U(\hat{x}+\hat{y}), U(\hat{x}+\hat{y})\rangle = \langle (\hat{x}+\hat{y}), (\hat{x}+\hat{y})\rangle$ we have $\langle U\hat{x}, U\hat{y}\rangle = \langle \hat{x}, \hat{y}\rangle$.

(b.) This says orthogonal transformations preserve angles as well as length.

16. (a.) $Let\ \ P(x) \in V.$ Then $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n a_n x^n$

so $P(x)$ is a linear combination of $\{1, x, \dots, x^n\}$.

If $P(x) = 0,$ then each $a_i = 0$.

$$T(1) = \frac{d}{dx}(1) = 0 = 0 \cdot 1 + 0x + 0x^2 + \cdots + 0x^n$$

$$T(x) = \frac{d}{dx}(x) = 1 = 1 \cdot 1 + 0x + 0x^2 + \cdots + 0x^n$$

$$T(x^2) = \frac{d}{dx}(x^2) = 2x = 0 = 0 \cdot 1 + 2x + 0x^2 + \cdots + 0x^n$$

$$\vdots$$

$$T(x^n) = \frac{d}{dx}(x^n) = nx^{n-1} = 0 = 0 \cdot 1 + 2x + 0x^2 + \cdots nx^{n-1} + 0x^n$$

so the matrix of T with respect to this basis is

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & n \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(c.) $T(V)$ is the vector space of polynomials with real coefficients of degree

$n-1$ or less. A basis is $\{1, x, \dots, x^{n-1}\}$. The dimension of $T(V)$ is n.

17.Suppose $T(\hat{x}) = \hat{0}$ and $T(\hat{y}) = \hat{0}$. Then $T(a\hat{x} + b\hat{y}) = aT(\hat{x}) + bT(\hat{y}) = a\hat{0} + b\hat{0} = \hat{0}$. The kernel of T for parts (a.) and (b.) is the constant polynomials.

Section 1-2

1. The area of the parallelogram formed by $\hat{B}$ and $\hat{C}$ is $\|\hat{B}\|\,\|\hat{C}\||\sin\theta|$ where θ is the angle formed by $\hat{B}$ and $\hat{C}$. See Figure 1. The volume of the parallelepiped formed by the non-coplanar vectors $\hat{A}, \hat{B}$ and $\hat{C}$ is the area of the parallelogram formed by $\hat{B}$ and $\hat{C}$ multiplied by the length of $\hat{A}$ and $\sin\varphi$ where φ is the angle $\hat{A}$ makes with the plane formed by $\hat{B}$ and $\hat{C}$. Now $\hat{B}\times\hat{C}$ is perpendicular to the plane formed by $\hat{B}$ and $\hat{C}$ so the volume of the parallelepiped is the area of the parallelogram formed by $\hat{B}$ and $\hat{C}$ multiplied by the length of $\hat{A}$ and cos $\propto$ where α is the angle $\hat{A}$ makes with $\hat{B}\times\hat{C}$. See Figure 1.

Now $\|\hat{B}\|\,\|\hat{C}\||\sin\theta| = \|\hat{B}\times\hat{C}\|$ so the volume of the parallelepiped is

$$\|\hat{B}\times\hat{C}\|\,\|\hat{A}\|\cos\varphi = \left|\hat{A}\cdot\left(\hat{B}\times\hat{C}\,\right)\right|.$$

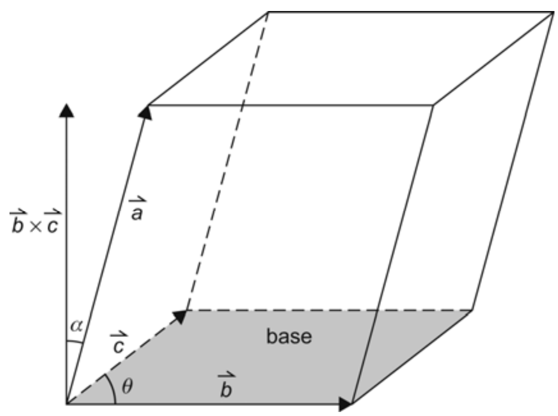

Figure 1

2.(a) Spherical coordinates:

$$x = r\cos\varphi\sin\theta\,,y = \; r\sin\varphi\sin\theta\,,z = r\sin\varphi\sin\theta$$

$$\hat{r} = r\cos\varphi\sin\theta\,\hat{\imath} + r\sin\varphi\sin\theta\,\hat{\jmath} + r\cos\theta\,\hat{k}.$$

Orthogonality:

$$\frac{\partial\hat{r}}{\partial\theta} = r\cos\varphi\cos\theta\,\hat{\imath} + r\sin\varphi\cos\theta\,\hat{\jmath} - r\sin\theta\,\hat{k}$$

$$\frac{\partial\hat{r}}{\partial\varphi} = -r\sin\varphi\sin\theta\,\hat{\imath} + r\cos\varphi\sin\theta\,\hat{\jmath}$$

$$\frac{\partial\hat{r}}{\partial r} = \cos\varphi\sin\theta\,\hat{\imath} + \sin\varphi\sin\theta\,\hat{\jmath} + \cos\theta\,\hat{k}$$

$$\langle\frac{\partial\hat{r}}{\partial\theta},\frac{\partial\hat{r}}{\partial\varphi}\rangle = r^2(-\cos\varphi\cos\theta\sin\varphi\sin\theta + \sin\varphi\cos\theta\cos\varphi\sin\theta) = 0$$

$$\langle\frac{\partial\hat{r}}{\partial\varphi},\frac{\partial\hat{r}}{\partial r}\rangle = r(-\sin\varphi\sin\theta\cos\varphi\sin\theta + \cos\varphi\sin\theta\sin\varphi\sin\theta) = 0$$

$$\langle\frac{\partial\hat{r}}{\partial\theta},\frac{\partial\hat{r}}{\partial r}\rangle = r(\cos\varphi\cos\theta\cos\varphi\sin\theta + \sin\varphi\cos\theta\sin\varphi\sin\theta - \sin\theta\cos\theta)$$
$$= r[(\cos^2\varphi + \sin^2\varphi)\cos\theta\sin\theta - \sin\theta\cos\theta] = 0.$$

Scaling factors:

$$h_1 = h_\theta = \left\|\frac{\partial\hat{r}}{\partial\theta}\right\| = \sqrt{(r\cos\varphi\cos\theta)^2 + (r\sin\varphi\cos\theta)^2 + (-r\sin\theta)^2}$$
$$= r\sqrt{\cos^2\theta(\cos^2\varphi + \sin^2\varphi) + \sin^2\theta} = r$$

$$h_2 = h_\varphi = \left\|\frac{\partial\hat{r}}{\partial\varphi}\right\| = \sqrt{(-r\sin\varphi\sin\theta)^2 + (r\cos\varphi\sin\theta)^2} = r\sin\theta$$

$$h_3 = h_r = \left\|\frac{\partial\hat{r}}{\partial r}\right\| = \sqrt{(\cos\varphi\sin\theta)^2 + (\sin\varphi\sin\theta)^2 + (\cos\theta)^2} = \sqrt{(\sin\theta)^2 + (\cos\theta)^2}$$
$$= 1.$$

Orthonormal Basis:

$$\hat{e}_1 = \hat{e}_\theta = \frac{\frac{\partial\hat{r}}{\partial\theta}}{h_1} = \frac{r\cos\varphi\cos\theta\,\hat{\imath} + r\sin\varphi\cos\theta\,\hat{\jmath} - r\sin\theta\,\hat{k}}{r}$$
$$= \cos\varphi\cos\theta\,\hat{\imath} + \sin\varphi\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{k}.$$

$$\hat{e}_2 = \hat{e}_\varphi = \frac{\frac{\partial\hat{r}}{\partial\varphi}}{h_2} = \frac{-r\sin\varphi\sin\theta\,\hat{\imath} + r\cos\varphi\sin\theta\,\hat{\jmath}}{r\sin\theta} = -r\sin\varphi\,\hat{\imath} + \cos\varphi\;\;\hat{\jmath}.$$

$$\hat{e}_3 = \hat{e}_r = \frac{\frac{\partial\hat{r}}{\partial r}}{h_3} = \frac{\cos\varphi\sin\theta\,\hat{\imath} + \sin\varphi\sin\theta\,\hat{\jmath} + \cos\theta\,\hat{k}}{1}.$$

Volume element:

$$dV = h_1h_2h_3du_1du_2du_3 = (r)(r\sin\theta)(1)d\theta d\varphi dr.$$

Gradient:

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{e}_3 = \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{e}_\varphi + \frac{\partial f}{\partial r}\hat{e}_r.$$

Laplacian:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right].$$

$h_1 h_2 h_3 = r^2 \sin\theta$,

$$\frac{h_2 h_3}{h_1} = \frac{r\sin\theta}{r} = \sin\theta, \quad \frac{h_1 h_3}{h_2} = \frac{r}{r\sin\theta} = \frac{1}{\sin\theta}, \quad \frac{h_1 h_2}{h_3} = r^2\sin\theta$$

$$\text{so } \nabla^2 f = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}\right) + \frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right)\right].$$

(b.) Parabolic cylindrical coordinates:

$$x = \xi\eta, y = \frac{1}{2}(\eta^2 - \xi^2), z = z$$

$$\hat{r} = \xi\eta\hat{\imath} + \frac{1}{2}(\eta^2 - \xi^2)\hat{\jmath} + z\hat{k}.$$

Orthogonality:

$$\frac{\partial \hat{r}}{\partial \xi} = \eta\hat{\imath} - \xi\hat{\jmath}$$

$$\frac{\partial \hat{r}}{\partial \eta} = \xi\hat{\imath} + \eta\hat{\jmath}$$

$$\frac{\partial \hat{r}}{\partial z} = \hat{k}$$

$$\langle\frac{\partial \hat{r}}{\partial \xi}, \frac{\partial \hat{r}}{\partial \eta}\rangle = \eta\xi - \xi\eta = 0; \quad \langle\frac{\partial \hat{r}}{\partial \xi}, \frac{\partial \hat{r}}{\partial z}\rangle = 0; \quad \langle\frac{\partial \hat{r}}{\partial \eta}, \frac{\partial \hat{r}}{\partial z}\rangle = 0.$$

Scaling factors:

$$h_1 = h_\xi = \left\|\frac{\partial \hat{r}}{\partial \xi}\right\| = \sqrt{\eta^2 + (-\xi)^2} = \sqrt{\eta^2 + \xi^2}$$

$$h_2 = h_\eta = \left\|\frac{\partial \hat{r}}{\partial \eta}\right\| = \sqrt{\xi^2 + \eta^2}$$

$$h_3 = h_z = \left\|\frac{\partial \hat{r}}{\partial r}\right\| = 1.$$

Orthonormal Basis:

$$\hat{e}_1 = \hat{e}_\xi = \frac{\frac{\partial \hat{r}}{\partial \xi}}{h_\xi} = \frac{\eta\hat{\imath} - \xi\hat{\jmath}}{\sqrt{\eta^2 + \xi^2}} = \frac{\eta}{\sqrt{\eta^2 + \xi^2}}\hat{\imath} - \frac{\xi}{\sqrt{\eta^2 + \xi^2}}\hat{\jmath}$$

$$\hat{e}_2 = \hat{e}_\eta = \frac{\frac{\partial \hat{r}}{\partial \eta}}{h_\eta} = \frac{\xi}{\sqrt{\eta^2 + \xi^2}}\hat{\imath} + \frac{\eta}{\sqrt{\eta^2 + \xi^2}}\ \hat{\jmath}$$

$$\hat{e}_3 = \hat{e}_z = \frac{\frac{\partial \hat{r}}{\partial r}}{h_z} = \hat{k}.$$

Volume element:

$$dV = h_1 h_2 h_3 du_1 du_2 du_3 = \left(\sqrt{\eta^2 + \xi^2}\right)\left(\sqrt{\eta^2 + \xi^2}\right)(1) d\xi d\eta dz.$$

Gradient:

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{e}_3 = \frac{1}{\sqrt{\eta^2 + \xi^2}}\frac{\partial f}{\partial \xi}\hat{e}_\xi + \frac{1}{\sqrt{\eta^2 + \xi^2}}\frac{\partial f}{\partial \eta}\hat{e}_\eta + \frac{\partial f}{\partial z}\hat{e}_z.$$

Laplacian:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right].$$

$$h_1 h_2 h_3 = \eta^2 + \xi^2, \qquad \frac{h_2 h_3}{h_1} = 1, \frac{h_1 h_3}{h_2} = 1, \quad \frac{h_1 h_2}{h_3} = \eta^2 + \xi^2$$

$$\text{so } \nabla^2 f = \frac{1}{\eta^2 + \xi^2}\left[\frac{\partial}{\partial \xi}\left(\frac{\partial f}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial f}{\partial \eta}\right) + \frac{\partial}{\partial z}\left((\eta^2 + \xi^2)\frac{\partial f}{\partial z}\right)\right]$$

$$= \frac{1}{\eta^2 + \xi^2}\frac{\partial^2 f}{\partial \xi^2} + \frac{1}{\eta^2 + \xi^2}\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial z^2}.$$

(c.) Parabolic coordinates:

$$x = \xi\eta\cos\varphi$$

$$y = \xi\eta\sin\varphi$$

$$z = \frac{1}{2}(\eta^2 - \xi^2)$$

$$\hat{r} = \xi\eta\cos\varphi\,\hat{\imath} + \xi\eta\sin\varphi\,\hat{\jmath} + \frac{1}{2}(\eta^2 - \xi^2)\hat{k}.$$

Orthogonality:

$$\frac{\partial\hat{r}}{\partial\xi} = \eta\cos\varphi\,\hat{\imath} + \eta\sin\varphi\,\hat{\jmath} - \xi\hat{k}$$

$$\frac{\partial\hat{r}}{\partial\eta} = \xi\cos\varphi\,\hat{\imath} + \xi\sin\varphi\,\hat{\jmath} + \eta\hat{k}$$

$$\frac{\partial\hat{r}}{\partial\varphi} = -\xi\eta\sin\varphi\,\hat{\imath} + \xi\eta\cos\varphi\,\hat{\jmath}$$

$$\langle\frac{\partial\hat{r}}{\partial\xi},\frac{\partial\hat{r}}{\partial\eta}\rangle = \eta\xi\cos^2\phi + \xi\eta\sin^2\phi - \xi\eta = 0$$

$$\langle\frac{\partial\hat{r}}{\partial\xi},\frac{\partial\hat{r}}{\partial\phi}\rangle = -\eta\cos\varphi(\xi\eta)\sin\varphi + \eta\sin\varphi(\xi\eta)\cos\varphi = 0$$

$$\langle\frac{\partial\hat{r}}{\partial\eta},\frac{\partial\hat{r}}{\partial\varphi}\rangle = \xi\cos\varphi\,(-\xi\eta\sin\varphi) + \xi\sin\varphi\,(\xi\eta)\cos\varphi = 0.$$

Scaling factors:

$$h_1 = h_\xi = \left\|\frac{\partial\hat{r}}{\partial\xi}\right\| = \sqrt{\eta^2(\cos^2\varphi + \sin^2\varphi) + (-\xi)^2} = \sqrt{\eta^2 + \xi^2}$$

$$h_2 = h_\eta = \left\|\frac{\partial\hat{r}}{\partial\eta}\right\| = \sqrt{\xi^2(\cos^2\varphi + \sin^2\varphi) + \eta^2} = \sqrt{\eta^2 + \xi^2}$$

$$h_3 = h_\varphi = \left\|\frac{\partial\hat{r}}{\partial\varphi}\right\| = \sqrt{(-\xi\eta\sin\varphi)^2 + (\xi\eta\cos\varphi)^2} = \eta\xi.$$

Orthonormal Basis:

$$\hat{e}_1 = \hat{e}_\xi = \frac{\frac{\partial\hat{r}}{\partial\xi}}{h_\xi} = \frac{\eta\cos\varphi\,\hat{\imath} + \eta\sin\varphi\,\hat{\jmath} - \xi\hat{k}}{\sqrt{\eta^2 + \xi^2}} = \frac{\eta\cos\varphi}{\sqrt{\eta^2 + \xi^2}}\hat{\imath} + \frac{\eta\sin\varphi}{\sqrt{\eta^2 + \xi^2}}\hat{\jmath} - \frac{\xi}{\sqrt{\eta^2 + \xi^2}}\hat{k}$$

$$\hat{e}_2 = \hat{e}_\eta = \frac{\frac{\partial \hat{r}}{\partial \eta}}{h_\eta} = \frac{\xi \cos\varphi\, \hat{\imath} + \xi \sin\varphi\, \hat{\jmath} + \eta \hat{k}}{\sqrt{\eta^2 + \xi^2}} = \frac{\xi \cos\varphi}{\sqrt{\eta^2 + \xi^2}}\hat{\imath} + \frac{\xi \sin\varphi}{\sqrt{\eta^2 + \xi^2}}\hat{\jmath} + \frac{\eta}{\sqrt{\eta^2 + \xi^2}}\hat{k}$$

$$\hat{e}_3 = \hat{e}_\varphi = \frac{\frac{\partial \hat{r}}{\partial \varphi}}{h_\varphi} = \frac{-\xi\eta \sin\varphi\ \ \hat{\imath} + \xi\eta \cos\varphi\, \hat{\jmath}}{\eta\xi} = -\sin\varphi\ \ \hat{\imath} + \cos\varphi\, \hat{\jmath}.$$

Volume element:

$$\begin{aligned} dV &= h_1 h_2 h_3 du_1 du_2 du_3 = \left(\sqrt{\eta^2 + \xi^2}\right)\left(\sqrt{\eta^2 + \xi^2}\right)(\eta\xi)d\xi d\eta d\varphi \\ &= (\eta^3\xi + \eta\xi^3)d\xi d\eta d\varphi. \end{aligned}$$

Gradient:

$$\begin{aligned} \nabla f &= \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{e}_3 \\ &= \frac{1}{\sqrt{\eta^2 + \xi^2}}\frac{\partial f}{\partial \xi}\hat{e}_\xi + \frac{1}{\sqrt{\eta^2 + \xi^2}}\frac{\partial f}{\partial \eta}\hat{e}_\eta + \frac{1}{\eta\xi}\frac{\partial f}{\partial \varphi}\hat{e}_\varphi. \end{aligned}$$

Laplacian:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right].$$

$$h_1 h_2 h_3 = (\eta^2 + \xi^2)\eta\xi, \qquad \frac{h_2 h_3}{h_1} = \eta\xi, \frac{h_1 h_3}{h_2} = \eta\xi, \quad \frac{h_1 h_2}{h_3} = \frac{\eta^2 + \xi^2}{\eta\xi}$$

$$\text{so } \nabla^2 f = \frac{1}{(\eta^2 + \xi^2)\eta\xi}\left[\frac{\partial}{\partial \xi}\left(\eta\xi\frac{\partial f}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\eta\xi\frac{\partial f}{\partial \eta}\right) + \frac{\partial}{\partial \varphi}\left(\left(\frac{\eta^2 + \xi^2}{\eta\xi}\right)\frac{\partial f}{\partial \varphi}\right)\right].$$

(d.) Bipolar coordinates:

$$x = \frac{a \sinh\eta}{\cosh\eta - \cos\xi}$$

$$y = \frac{a \sin\xi}{\cosh\eta - \cos\xi}$$

$$z = z.$$

$$\hat{r} = \frac{a \sinh \eta}{\cosh \eta - \cos \xi}\hat{\imath} + \frac{a \sin \xi}{\cosh \eta - \cos \xi}\hat{\jmath} + z\hat{k}.$$

We set $a = 1$ to simplify the notation.

Orthogonality:

Note that

$$\frac{\partial x}{\partial \eta} = \frac{(\cosh \eta - \cos \xi) \cosh \eta - \sinh \eta \sinh \eta}{(\cosh \eta - \cos \xi)^2} = \frac{1 - \cos \xi \cosh \eta}{(\cosh \eta - \cos \xi)^2}$$

$$\frac{\partial x}{\partial \xi} = \frac{-\sin \xi \sinh \eta}{(\cosh \eta - \cos \xi)^2}$$

$$\frac{\partial y}{\partial \eta} = \frac{-\sin \xi \sinh \eta}{(\cosh \eta - \cos \xi)^2}$$

$$\frac{\partial y}{\partial \xi} = \frac{(\cosh \eta - \cos \xi) \cos \xi - \sin \xi \sin \xi}{(\cosh \eta - \cos \xi)^2} = \frac{\cosh \eta \cos \xi - 1}{(\cosh \eta - \cos \xi)^2}.$$

$$\frac{\partial \hat{r}}{\partial \xi} = \frac{-\sin \xi \sinh \eta}{(\cosh \eta - \cos \xi)^2}\hat{\imath} + \frac{\cosh \eta \cos \xi - 1}{(\cosh \eta - \cos \xi)^2}\hat{\jmath}$$

$$\frac{\partial \hat{r}}{\partial \eta} = \frac{1 - \cos \xi \cosh \eta}{(\cosh \eta - \cos \xi)^2}\hat{\imath} + \frac{-\sin \xi \sinh \eta}{(\cosh \eta - \cos \xi)^2}\hat{\jmath}$$

$$\frac{\partial \hat{r}}{\partial \varphi} = \hat{k}.$$

$$\begin{aligned}\langle \frac{\partial \hat{r}}{\partial \xi}, \frac{\partial \hat{r}}{\partial \eta} \rangle \\ &= \left[\frac{-\sin \xi \sinh \eta}{(\cosh \eta - \cos \xi)^2}\right]\left[\frac{1 - \cos \xi \cosh \eta}{(\cosh \eta - \cos \xi)^2}\right] \\ &+ \left[\frac{\cosh \eta \cos \xi - 1}{(\cosh \eta - \cos \xi)^2}\right]\left[\frac{-\sin \xi \sinh \eta}{(\cosh \eta - \cos \xi)^2}\right] = 0\end{aligned}$$

$$\langle \frac{\partial \hat{r}}{\partial \xi}, \frac{\partial \hat{r}}{\partial z} \rangle = 0$$

$$\langle \frac{\partial \hat{r}}{\partial \eta}, \frac{\partial \hat{r}}{\partial z} \rangle = 0.$$

Scaling factors:

$$h_1 = h_\xi = \left\| \frac{\partial \hat{r}}{\partial \xi} \right\| = \sqrt{\left[\frac{-\sin\xi \sinh\eta}{(\cosh\eta - \cos\xi)^2} \right]^2 + \left[\frac{\cosh\eta\cos\xi - 1}{(\cosh\eta - \cos\xi)^2} \right]^2}$$

Now

$$\begin{aligned}(-\sin\xi\sinh\eta)^2 + (\cosh\eta\cos\xi - 1)^2 &= (\cosh\eta\cos\xi - 1)^2 + \sin^2\xi\sinh^2\eta \\ &= (\cosh\eta\cos\xi - 1)^2 + \sin^2\xi(\cosh^2\eta - 1) \\ &= 1 - 2\cosh\eta\cos\xi + cos^2\xi\cosh^2\eta + \sin^2\xi\cosh^2\eta - \sin^2\xi \\ &= 1 - 2\cosh\eta\cos\xi + \cosh^2\eta - \sin^2\xi \\ &= \cos^2\xi - 2\cosh\eta\cos\xi + \cosh^2\eta = (\cosh\eta - \cos\xi)^2.\end{aligned}$$

Thus

$$h_\xi = \left\| \frac{\partial \hat{r}}{\partial \xi} \right\| = \frac{(\cosh\eta - \cos\xi)}{(\cosh\eta - \cos\xi)^2} = \frac{1}{\cosh\eta - \cos\xi}.$$

Likewise

$$h_2 = h_\eta = \left\| \frac{\partial \hat{r}}{\partial \eta} \right\| = \sqrt{\left[\frac{1 - \cos\xi\cosh\eta}{(\cosh\eta - \cos\xi)^2} \right]^2 + \left[\frac{\cosh\eta\cos\xi - 1}{(\cosh\eta - \cos\xi)^2} \right]^2} = \frac{1}{\cosh\eta - \cos\xi}$$

$$h_3 = h_z = \left\| \frac{\partial \hat{r}}{\partial z} \right\| = 1.$$

$$\text{In the case } a \neq 1, h_\xi = h_\eta = \frac{a}{\cosh\eta - \cos\xi}.$$

Orthonormal Basis:

$$\hat{e}_1 = \hat{e}_\xi = \frac{\frac{\partial \hat{r}}{\partial \xi}}{h_\xi} = \frac{\frac{-\sin\xi\sinh\eta}{(\cosh\eta - \cos\xi)^2}\hat{\imath} + \frac{\cosh\eta\cos\xi - 1}{(\cosh\eta - \cos\xi)^2}\hat{\jmath}}{\frac{1}{\cosh\eta - \cos\xi}}$$

$$= \frac{-\sin\xi \sinh\eta}{(\cosh\eta - \cos\xi)} \hat{\imath} + \frac{\cosh\eta \cos\xi - 1}{(\cosh\eta - \cos\xi)} \hat{\jmath}$$

$$\hat{e}_2 = \hat{e}_\eta = \frac{\frac{\partial \hat{r}}{\partial \eta}}{h_\eta} = \frac{\frac{1 - \cos\xi \cosh\eta}{(\cosh\eta - \cos\xi)^2} \hat{\imath} + \frac{-\sin\xi \sinh\eta}{(\cosh\eta - \cos\xi)^2} \hat{\jmath}}{\frac{1}{\cosh\eta - \cos\xi}}$$

$$= \frac{1 - \cos\xi \cosh\eta}{(\cosh\eta - \cos\xi)} \hat{\imath} - \frac{\sin\xi \sinh\eta}{(\cosh\eta - \cos\xi)} \hat{\jmath}$$

$$\hat{e}_3 = \hat{e}_z = \frac{\frac{\partial \hat{r}}{\partial z}}{h_z} = \hat{k}.$$

Volume element:

$$dV = h_1 h_2 h_3 du_1 du_2 du_3 = \frac{a^2}{(\cosh\eta - \cos\xi)^2} d\xi d\eta dz.$$

Gradient:

$$\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \hat{e}_3$$

$$= \frac{(\cosh\eta - \cos\xi)}{a} \frac{\partial f}{\partial \xi} \hat{e}_\xi + \frac{(\cosh\eta - \cos\xi)}{a} \frac{\partial f}{\partial \eta} \hat{e}_\eta + \frac{\partial f}{\partial z} \hat{e}_z.$$

Laplacian:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial u_3} \right) \right].$$

$$h_1 h_2 h_3 = \frac{a^2}{(\cosh\eta - \cos\xi)^2}, \qquad \frac{h_2 h_3}{h_1} = 1, \frac{h_1 h_3}{h_2} = 1, \quad \frac{h_1 h_2}{h_3} = \frac{a^2}{(\cosh\eta - \cos\xi)^2}$$

so $\nabla^2 f$

$$= \frac{(\cosh\eta - \cos\xi)^2}{a^2} \left[\frac{\partial}{\partial \xi} \left(\frac{\partial f}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial f}{\partial \eta} \right) + \frac{\partial}{\partial z} \left(\left(\frac{a^2}{(\cosh\eta - \cos\xi)^2} \right) \frac{\partial f}{\partial z} \right) \right].$$

(e.) Prolate spheriodal coordinates: