Chapter 3, Section 1
24 July 2001

Exercise 3.1

$M = 20\text{ kg},\ \omega_{nat} = 100\text{ rad/s} \Rightarrow k = M\omega_{nat}^2 = 2(10^5)\text{ N/m}$

$q = 0.02\cos(110t - 1.5)\text{ meter}$

$= \mathrm{Re}\left[0.02\, e^{i(\omega t - 1.5)}\right] \quad \omega = 110\text{ rad/s}$

$Q(t) = M\ddot{q} + C\dot{q} + Kq$

$= \mathrm{Re}\left[(-M\omega^2 + Ci\omega + K)(0.02)e^{i(\omega t - 1.5)}\right]$

But $\frac{K}{M} = \omega_{nat}^2$ & $C/M = 2\zeta\omega_{nat}$, so

$Q(t) = M\,\mathrm{Re}\left[(-\omega^2 + 2i\zeta\omega_{nat}\omega + \omega_{nat}^2)(0.02)\right.$
$\left.\times\, e^{i(\omega t - 1.5)}\right]$

For $\zeta = 0$ & $\omega = 110$:

$Q = 20\,\mathrm{Re}\left[-42\, e^{i(110t - 1.5)}\right]$

$\Rightarrow \quad = -840\cos(110t - 1.5)$ newton

For $\zeta = 0.4$ & $\omega = 110$ rad/s:

$Q = 20\,\mathrm{Re}\left[(-2100 + 8800i)(0.02)\, e^{i(110t - 1.5)}\right]$

$= \mathrm{Re}\left[(-840 + 3520i)\, e^{i(110t - 1.5)}\right]$

$= \mathrm{Re}\left[3619\, e^{i\,1.8051}\, e^{i(110t - 1.5)}\right]$

$\Rightarrow \quad = 3619\cos(110t + 0.3051)$ newton

Exercise 3.2

$$m\ddot{q} + c\dot{q} + kq = F\cos(\omega t)$$

$m = 8$ kg, $\zeta = 0.25$, $\omega_d = 10\pi$ rad/s, $F/k = 0.002$ meter

$$\omega_{nat} = \frac{\omega d}{(1-\zeta^2)^{1/2}} = 32.446 \text{ rad/s}$$

$$r = \frac{5.2(2\pi)}{\omega_{nat}} = 1.006976$$

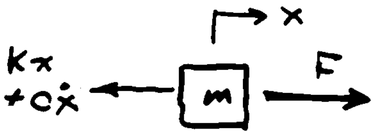
$$q = \frac{F}{k}|D(r,\zeta)|\cos(\omega t - \phi)$$

$$|D(r,\zeta)| = \frac{1}{[(1-r^2)^2 + 4\zeta^2 r^2]^{1/2}} = 1.9854$$

$$\phi = \tan^{-1}\frac{2\zeta r}{(1-r^2)} = -1.5430 + \pi = 1.5986 \text{ rad} = 91.593^\circ$$

Thus $q = 0.003971\cos(10.4\pi t - 1.5986)$ meter

Exercise 3.3



$$M\ddot{x} + C\dot{x} + Kx = Re\{F\exp(i\omega t)\}$$

$$M = 80\,kg,\ \omega_{nat} = 12.2(2\pi)\ rad/s$$

$$K = M\omega_{nat}^2 = 4.701(10^5)\ N/m$$

$$\omega = 12(2\pi)\ rad/s \Rightarrow r = \frac{12}{12.2}$$

$$X = \frac{F}{k}\,\frac{1}{1 - r^2 + 2i\zeta r}$$

Then $|\ddot{x}| = \omega^2|x| = \dfrac{|F|}{K}\,\dfrac{\omega^2}{|1 - r^2 + 2i\zeta r|} < 15g$

so $|F| < \dfrac{15gK}{\omega^2}\left|(1 - r^2 + 2i\zeta r)\right|$

$\zeta = 0 \Rightarrow |F| < 395.6\ N$

$\zeta = 0.05 \Rightarrow F < |395.6 + 1196.51i| = 1260.2\ N$

Exercise 3.4

Given $K = 5000\,N/m$, $M = 2\,kg$, $F = 20 \sin(\omega t)\,N$ gives

$q = -0.010 \cos(\omega t)$ meter.

Find ζ & ω

Solution: Write the force & response in complex form;

$$F = Re\{F e^{i\omega t}\}, \quad q = Re\{X e^{i\omega t}\}$$

where $F = \frac{20}{i}$ & $X = -0.010$

but $X = \frac{F}{k} D(r,\zeta) \Rightarrow D(r,\zeta) = \frac{kX}{F} = -2.5i = 25e^{-i\pi/2}$

but $D(r,\zeta) = |D| e^{-i\phi}$ so

$|D| = 20$ & $\phi = \pi/2$

Whenever $\phi = \pi/2$, it must be that $r = 1$

Thus $\omega = \omega_{nat} = \left(\frac{K}{M}\right)^{1/2} = 50$ rad/s $\Leftarrow$

Then, at $r = 1$, $|D| = \frac{1}{2\zeta} \Rightarrow \zeta = \frac{1}{2(2.5)} = 0.20 \Leftarrow$

Exercise 3.5

Given $m = 10$ kg, $\omega_{nat} = 1000(2\pi)$ rad/s, $Q = 1200\sin(\omega t)$ newton, $|q| = 0.0024$ meter at $\omega = \omega_{nat}$. Find $q(t)$ when $\omega = 0.95\,\omega_{nat}$ & $\omega = 1.05\,\omega_{nat}$

Solution: Steady state response in general is

$$Q(t) = Re\{F e^{i\omega t}\} \Rightarrow q = Re\left\{\frac{F}{k}|D|\, e^{i(\omega t - \phi)}\right\}$$

The given force is $Q = 1200\sin(\omega t) = Re\left\{\frac{1200}{i} e^{i\omega t}\right\}$

Thus $F = \frac{1200}{i}$

To find k, use $\omega_{nat} = \left(\frac{k}{M}\right)^{1/2} \Rightarrow k = \omega_{nat}^2 m = 3.948(10^8)$ N/m

Also when $\omega = \omega_{nat}$, $r = 1 \Rightarrow D = \frac{1}{1 - r^2 + 2i\zeta r} = \frac{1}{2i\zeta}$

Thus the amplitude at $\omega = \omega_{nat}$ is

$$|q| = 0.0024 = \frac{|F|}{k}|D| = \frac{1200}{3.948(10^8)}\ \frac{1}{2\zeta}$$

Solve: $\zeta = 6.333(10^{-4})$

At any r: $|D| = \frac{1}{[(1-r^2)^2 + 4\zeta^2 r^2]^{1/2}}$ & $\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$

When $\omega = 0.95\,\omega_{nat} \Rightarrow r = 0.95$

$|D| = 10.256$ & $\phi = 0.01234$ rad $= 0.707°$

$$q = Re\left\{\frac{1200/i}{k}(10.256)\, e^{i(\omega t - \phi)}\right\}$$

$$= Re\left\{\frac{3.117(10^{-5})}{i} e^{i(\omega t - 0.1234)}\right\}$$

$$= 3.117(10^{-5})\sin(190\pi t - 0.01234) \text{ meter} \Leftarrow$$

When $\omega = 1050(2\pi)$ rad/s $\Rightarrow r = 1.05$

$|D| = 9.755$ & $\phi = 3.129$ rad $= 179.26°$ (note $0 < \phi < \pi$ rad)

$$q = Re\left\{\frac{1200(9.755)}{i\,k} e^{i(\omega t - 3.129)}\right\}$$

$$= 2.965(10^{-5})\sin(210\pi t - 3.129) \text{ meter} \Leftarrow$$

Exercise 3.6

Given $F = 60\sin(\omega t) + 80\cos(\omega t) \Rightarrow |q| = 0.05$ meter @ $\omega = 0$,

$\omega_{nat} = 20(2\pi)$ rad/s, $q = X\sin(\omega t)$ at $\omega = 18(2\pi)$ rad/s

Find X

Express the force as a complex variable:

$F(t) = \mathrm{Re}\{(\frac{60}{i} + 80)e^{i\omega t} = \mathrm{Re}(100e^{-0.6435i}e^{i\omega t})$

Then $x = \mathrm{Re}(Xe^{i\omega t})$

where $X = \frac{F}{k} D(r,\zeta)$, $r = \omega/\omega_{nat}$

At $r = \frac{18}{20}$, $q = X\sin(\omega t) = \mathrm{Re}(\frac{|X|}{i}e^{i\omega t})$

At $r = 0$, $|q| = 0.05 = \frac{|F|}{k}|D|$ but $|D| = 1$ at $r = 0$

Thus $k = \frac{|F|}{0.05} = 2000$ N/m

Then $r = 0.9$ gives

$$\frac{|X|}{i} = \frac{100e^{-0.6435i}}{2000} D(r,\zeta) = 0.05e^{-0.6435i}|D|e^{-i\phi}$$

Match the arguments of the terms on either side,

using $\frac{1}{i} = e^{-i\pi/2} \Rightarrow -\frac{\pi}{2} = -0.6433 - \phi$

so $\phi = \frac{\pi}{2} - 0.6433 = 0.9275$

But $\tan\phi = \frac{2\zeta r}{1-r^2} \Rightarrow \zeta = \left[\frac{1-0.9^2}{2(0.9)}\right]\tan\phi = 0.1408$

Then $|D| = \frac{1}{[(1-0.9^2)^2 + 4\zeta^2(0.9^2)]^{1/2}} = 3.157$

$|X| = 0.05(3.157) = 0.1579$ meter $\Leftarrow$

Exercise 3.7

Given $Q(t) = 400\cos(\omega t)$, steady-state properties

1. $\omega = 40(2\pi)$ rad/s $\Rightarrow \dot{q} = \beta Q = \beta(400)\cos(80\pi t)$
2. $|q| = 0.016$ meter @ $\omega = 25(2\pi)$ rad/s
3. $\omega = 42(2\pi)$ rad/s $\Rightarrow$ graph of q vs t

Find ω_{nat}, ζ, k, c, max q in graph, $\dot{q}$ @ $\omega = 40(2\pi)$ rad/s

Solution:

From 1; Integrate $q = \dfrac{\beta(400)}{80\pi}\sin(80\pi t)$

The phase lag is $\phi = \pi/2$ because

$$Q = Re\{400 e^{i80\pi t}\}$$

$$q = Re\left\{\frac{\beta 400}{80\pi}\frac{1}{i}e^{i80\pi t}\right\}$$

But $\phi = \pi/2$ means $\omega = \omega_{nat}$ for any ζ.

Thus $\omega_{nat} = 80\pi$ rad/s $= 40$ Hz ⟸

From 2: $r = \dfrac{25}{\omega_{nat}} = \dfrac{25}{40} = 0.625$

$$q = Re\left\{\frac{F}{k}D(r,\zeta)e^{i\omega t}\right\}$$

$$|q| = 0.016 = \frac{|F|}{K}|D(r,\zeta)| = \frac{400}{K}\frac{1}{\left[(1-r^2)^2 + 4\zeta^2 r^2\right]^{1/2}}$$

From 3: $r = \dfrac{42}{\omega_{nat}} = \dfrac{42}{40} = 1.05$

In the graph let $q = |q|\cos(\omega t - \phi)$

Set $q = 0$ @ $t = 0.0025$ sec $\Rightarrow \omega t - \phi = \pm\pi/2, \pm 3\pi/2$

Because $\dot{q} = -\omega|q|\sin(\omega t - \phi) > 0$, $\omega t - \phi = -\pi/2$ or $3\pi/2$

This gives $\phi = 84\pi(0.0025) + \frac{\pi}{2}$ or $84\pi(0.0025) - 3\pi/2$

However, $0 < \phi < \pi \Rightarrow \phi = 2.231$

but $\tan(\phi) = \dfrac{2\zeta r}{1 - r^2} \Rightarrow \tan(2.231) = \dfrac{2\zeta(1.05)}{1 - 1.05^2}$

Solve: $\zeta = 0.06290$ ⟸

To find K go back to the 2nd fact

$$0.016 = \frac{400}{k}\frac{1}{[(1-r^2)^2+4\varsigma^2r^2]^{1/2}} \quad \text{with } r=0.625$$

Thus

$$k = \frac{400}{0.016}\frac{1}{[(1-0.625^2)^2+4\varsigma^2]^{1/2}}$$

$$= 4.069(10^4)\,N/m \Leftarrow$$

Next evaluate M using $\omega_{nat}^2 = \frac{k}{M}$,

so $M = \frac{K}{(80\pi)^2} = 0.6442$ kg $\Leftarrow$

Then $\frac{c}{M} = 2\varsigma\omega_{nat}$ gives

$$C = 0.6442(2)(0.6290)(80\pi) = 26.37 \text{ N-m/s} \Leftarrow$$

To find $|q|$ in the graph, set $r = \frac{42}{40} = 1.05$

$$|q| = \frac{|F|}{k}|D(r,\varsigma)|$$

$$= \frac{400}{4.069(10^4)}\frac{1}{[(1-1.05^2)^2+4\varsigma^2(1.05^2)]^{1/2}}$$

$$= 0.00588 \text{ meter} \Leftarrow$$

To find $\dot{q}$ at $r=1$ $(\omega=\omega_{nat})$

$$q = Re\left\{\frac{F}{k}D(r,\varsigma)e^{i\omega t}\right\}$$

so $$\dot{q} = Re\left\{i\omega\frac{F}{k}D(r,\varsigma)e^{i\omega t}\right\}$$

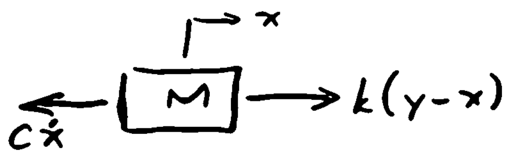
Set $r=1 \Rightarrow D(1,\varsigma) = \frac{1}{2\varsigma i}$

$$\dot{q} = Re\left\{\omega\frac{F}{k}\frac{1}{2\varsigma}e^{i\omega t}\right\}$$

$$= (80\pi)\frac{400}{k}\frac{1}{2\varsigma}Re\{e^{i\omega t}\}$$

$$= 19.63\cos(80\pi t) \text{ m/s} \Leftarrow$$

Exercise 3.8



$$M\ddot{x} + c\dot{x} + kx = ky$$

$$y = Re\left(\frac{Y}{i}\exp(i\omega t)\right)$$

Let $x = Re\left(\frac{X}{i}\exp(i\omega t)\right)$

so $$X = \frac{kY}{k + i\omega c - \omega^2 M}$$

$$x = Re\left[\frac{Y}{i}\;\frac{k}{k+i\omega c-\omega^2 M}\exp(i\omega t)\right]$$

$$F = k(y-x) = Re\left[\frac{Y}{i}\left(1 - \frac{k}{k+i\omega c-\omega^2 M}\right)\exp(i\omega t)\right]$$

$$= Re\left[\frac{Y}{i}\;\frac{i\omega c-\omega^2 M}{k+i\omega c-\omega^2 M}\exp(i\omega t)\right]$$

Rewrite x as

$$x = Re\left[\frac{Y}{i}\;\frac{k/M}{k/M + i\omega c/M - \omega^2}\exp(i\omega t)\right]$$

$$= Re\left[\frac{Y}{i}\;\frac{1}{1+2i\zeta r - r^2}\exp(i\omega t)\right], \quad r = 1.1$$

Thus $$|X| = |Y|\frac{1}{|1+2i\zeta r - r^2|}$$

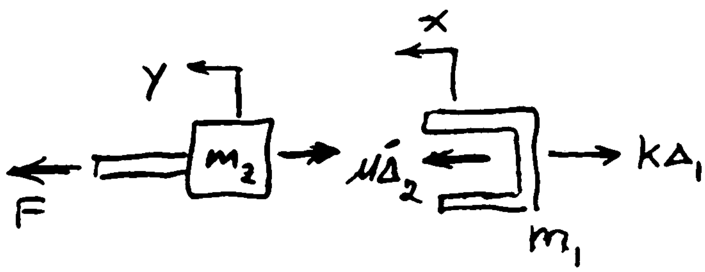
Set $r = 1.10$ & $\zeta = 0.2 \Rightarrow |X| = 2.051\,Y$ $\Leftarrow$

$$\arg(X) = \arg(Y) - \phi$$

$$= \arg(Y) - \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

$$= \arg(Y) - 2.935 \text{ rad}$$ $\Leftarrow$

Exercise 3.9



Given $y = A\sin\omega t$,
$m_1 = 0.5$ kg,
$m_2 = 1.0$ kg,
$k = 3.2(10^3)$ N/m,
$\mu = 40$ N-s/m,
$A = 0.02$ meter

Find amplitude & phase of F relative to y for $\omega = 150\pi$ & 170π.

Solution: Eqs of motion: $\Delta_1 = x$, $\Delta_2 = y - x$

Piston: $F - \mu(\dot{y} - \dot{x}) = m_2 \ddot{y}$

Tube: $\mu(\dot{y} - \dot{x}) - kx = m_1 \ddot{x}$

It is given that $y = A\sin(\omega t) = \mathrm{Re}\left[\frac{A}{i} e^{i\omega t}\right]$

Harmonic variation $\Rightarrow$ Assume $x = \mathrm{Re}\left[\frac{X}{i} e^{i\omega t}\right]$, $F = \mathrm{Re}\left[\frac{\hat{F}}{i} e^{i\omega t}\right]$

Note: Using $\frac{1}{i}$ as a factor for x & F means that the polar angles of $\hat{F}$ and X are phase angles relative to y.

Substitute into eqs of motion & cancel $e^{i\omega t}$:

$$\hat{F} - \mu(i\omega)(A - X) = -m_2 \omega^2 A \qquad (1)$$

$$\mu(i\omega)(A - X) - kX = -m_1 \omega^2 X \qquad (2)$$

These are two simultaneous equations for F and X at any ω because $A = 0.02$:

From (2): $$X = \frac{i\omega\mu}{k + i\omega\mu - m_1\omega^2} A$$

From (1):
$$\hat{F} = \left[(i\omega\mu - m_2\omega^2) + \frac{\mu^2\omega^2}{k + i\omega\mu - m_1\omega^2}\right] A$$

For $\omega = 75$ rad/s:

$F = -104.88 + 0.99i = 104.88 e^{i3.132}$

or $|F| = 104.88$ N @ $\phi = 3.132 \frac{180}{\pi} = 179.46°$ ahead of y $\Leftarrow$

For $\omega = 85$ rad/s:

$F = -152.63 + 0.99i = 152.63 e^{3.135i}$

or $|F| = 152.63$ N @ $\phi = 3.135 \frac{180}{\pi} = 179.63°$ ahead of y $\Leftarrow$

Exercise 3.10

Given plot of $|\theta|$ vs ω, $I_0 = 4$ kg-m^2

Estimate ω_{nat} & ζ. Then find K, C, & $|\Gamma|$ for system.

Solution: Because rotation $\Rightarrow \theta = q$, the eq of motion is

$$M\ddot{\theta} + C\dot{\theta} + K\theta = Re(\Gamma e^{i\omega t})$$

where $M = I_0$ for rotation.

Estimate ω_{nat} as the value of ω at the peak:

$$\omega_{nat} \doteq 35 \text{ Hz} = 70\pi \text{ rad/s}$$

Two ways to estimate ζ:

(a) At $\omega = 0$, $|\theta| = \frac{|F|}{k} \approx 0.30\left(\frac{\pi}{180}\right)$
at $\omega = \omega_{nat}$, $|\theta| = \frac{|F|}{k}\frac{1}{2\zeta} \approx 2.70\left(\frac{\pi}{180}\right)$

$$\zeta = \frac{0.30}{2(2.70)} \approx 0.056$$

(b) Bandwidth method:

Half power points $\omega_1 = 33(2\pi)$, $\omega_2 = 37(2\pi)$ rad/s

so $\Delta\omega = \omega_2 - \omega_1 = 4(2\pi)$ rad/s

$$QF = \frac{\omega_{nat}}{\Delta\omega} = \frac{35}{4} = \frac{1}{2\zeta} \Rightarrow \zeta = \frac{4}{2(35)} = 0.057$$

Both values agree. Use $\zeta = 0.057$

Then $\omega_{nat}^2 = \frac{K}{I_0} \Rightarrow K = I_0\omega_{nat}^2 = 1.934(10^5)$ N/m $\Leftarrow$

$$\zeta = \frac{C}{2(KI_0)^{1/2}} \Rightarrow C = 2\zeta(KI_0)^{1/2} = 100.3 \text{ N-s/m} \Leftarrow$$

At $\omega = 0$: $\frac{|F|}{k} = 0.30\left(\frac{\pi}{180}\right)$ & $|F|$ is the torque $|\Gamma|$ (rotation)

so $|\Gamma| = 0.30\left(\frac{\pi}{180}\right)K = 1012.9$ N-m $\Leftarrow$

Exercise 3.11

Given $Q(t) = F\cos(\omega t)$, $x = 4\sin(\omega t)$ at $\omega = 100(2\pi)$ rad/s,

$|q| = 2.83$ mm @ $\omega = 105(2\pi)$ rad/s

Find ϕ at $\omega = 105(2\pi)$, $|q|$ & ϕ at $\omega = 110(2\pi)$ rad/s

Solution: At $\omega = 200\pi$, $x = 4\sin(\omega t) \equiv 4\cos(\omega t - \frac{\pi}{2})$

so $\phi = \frac{\pi}{2}$ at $\omega = 200\pi \Rightarrow \omega = \omega_{nat} = 200\pi$ rad/s

In general $|q| = \frac{F}{k}|D(r,\zeta)|$

At $\omega = \omega_{nat}$, $|D| = \frac{1}{2\zeta} \Rightarrow 4 = \frac{F}{k}\frac{1}{2\zeta}$

At $\omega = 105(2\pi)$, $r = \frac{105}{100}$ & $|q| = 2.83 = \frac{F}{k}|D(1.05,\zeta)|$

Thus $\dfrac{|X(r=1.05)|}{|X(r=1)|} = \dfrac{2.83}{4} = 0.7075 \approx \dfrac{1}{\sqrt{2}}$

so $\omega = 105(2\pi)$ is the upper half power point

Bandwidth $\Delta\omega = 2(210\pi - 200\pi) = 20\pi$

$QF = \dfrac{\omega_{nat}}{\Delta\omega} = \dfrac{200\pi}{20\pi} = 10 = \dfrac{1}{2\zeta} \Rightarrow \zeta = \dfrac{1}{20} = 0.05$

Then $4 = \dfrac{F}{k}\dfrac{1}{2\zeta} \Rightarrow \dfrac{F}{k} = \dfrac{4}{10} = 0.4$

$|q| = \dfrac{F}{k}|D(r,\zeta)|$ & $\phi = \tan^{-1}\left(\dfrac{2\zeta r}{1-r^2}\right)$

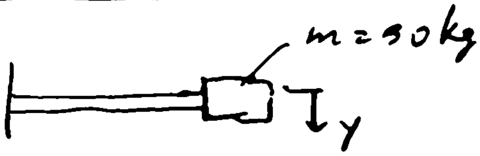
For $r = 1.05$, $\phi = -0.7974$ rad $+ \pi = 134.3°$ ⇐

For $\omega = 110(2\pi)$. $r = 1.1$:

$|D| = 4.218$ & $\phi = -0.4285 + \pi = 2.713$ rad $= 152.4°$ ⇐

so $|q| = \dfrac{F}{k}|D| = 0.4(4.218) = 1.687$ mm

Exercise 3.12



$F = mg \Rightarrow y = 0.004$ meter

$Q(t) = F_0 \sin(\omega t)$

$\omega = 0 \Rightarrow |y| = 0.02$ meter, $\omega = \omega_{nat} \Rightarrow |y| = 0.4$ meter

Static $y = \frac{mg}{K} = 0.004 \Rightarrow \omega_{nat}^2 = \frac{k}{m} = \frac{g}{0.004}$

$\omega_{nat} = \left(\frac{9.807}{0.004}\right)^{1/2} = 49.52$ rad/s

Then $\omega \approx 0 \Rightarrow |y| = 0.02 = \frac{F_0}{k}$

$\omega = \omega_{nat} \Rightarrow |y| = 0.4 = \frac{F_0}{k}\frac{1}{2\zeta} \Rightarrow 0.4 = 0.02\left(\frac{1}{2\zeta}\right)$

$\zeta = 0.025$

Assume viscous damping $\Rightarrow E_{dis} = \pi\omega C|y|^2$

$(P_{dis})_{av} = \frac{E_{dis}}{T} = \frac{1}{2}\omega^2 C|y|^2$ & $C = 2\zeta\omega_{nat} M$

$(P_{dis})_{av} = \zeta\omega_{nat}\omega^2 M|y|^2$

At any ω: $|y| = \frac{F_0}{K}|D(r,\zeta)|$

$(P_{dis})_{av} = \zeta\omega_{nat}\omega^2 M\left(\frac{F_0}{K}\right)^2|D(r,\zeta)|^2$

Set $\omega = r\,\omega_{nat} \Rightarrow (P_{dis})_{av} = \zeta\,\omega_{nat}^3 M\left(\frac{F_0}{k}\right)^2[r D(r,\zeta)]^2$

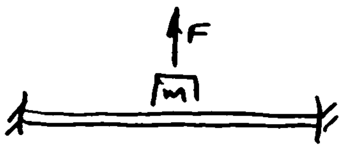
$P_{dis} = 0.025(49.52)^3(50)(0.02)^2[r D(r,\zeta)]^2$

$\Rightarrow \quad = 60.70 \dfrac{r^2}{(1-r^2)^2 + 4(0.025)^2 r^2}$ watts

$\Rightarrow \quad QF = \frac{\omega_{nat}}{\Delta\omega} = \frac{1}{2\zeta} \Rightarrow \Delta\omega = 2\zeta\omega_{nat} = 2.476$ rad/s $= 0.39$ Hz

$\Rightarrow \quad$ At $r = 0.9$, $(P_{dis})_{av} = 1290$ watt

Exercise 3.13



$m = 40 \text{ kg}, \quad q_{static} = \frac{mg}{k}$

$\therefore \quad \frac{k}{m} = \omega_{nat}^2 = \frac{g}{q_{static}} = \frac{9.807}{0.005}$

$\omega_{nat} = 44.288 \text{ rad/s}$

(a) Structural damping

$F = Re\{F_0 \exp(i\omega t)\}$ & $q = Re\{X \exp(i\omega t)\}$

$$X = \frac{F}{k} \frac{1}{1 - \left(\frac{\omega}{\omega_{nat}}\right)^2 + i\gamma} \Rightarrow |X| = \frac{F}{k} \frac{1}{\left\{\left[1 - \left(\frac{\omega}{\omega_{nat}}\right)^2\right]^2 + \gamma^2\right\}^{1/2}}$$

Max $|X| = 0.06$ meter $\Rightarrow \omega = \omega_{nat}$

$0.06 = \frac{|F|}{k} \frac{1}{\gamma}$ but $k = m\omega_{nat}^2 = 78456$ N/m

Set $|F| = 240 \Rightarrow \gamma = \frac{240}{0.06k} = 0.05098 \quad \Leftarrow$

For $\omega = 9(2\pi)$ rad/s $\Rightarrow \omega/\omega_{nat} = 1.2768 = r$

$$|X| = \frac{24}{k} \frac{1}{[(1-r^2)^2 + \gamma^2]^{1/2}} = 0.00484 = 4.84 \text{ mm} \quad \Leftarrow$$

(b) Viscous damping:

When $\omega/\omega_{nat} = 1 \Rightarrow |X| = \frac{|F|}{k} \frac{1}{2\zeta}$

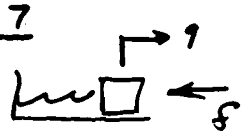
Thus $\zeta = \frac{240}{(0.06k)(2)} = 0.02549$

For $\omega/\omega_{nat} = 1.2768 = r$; $|X| = \frac{240}{k} \frac{1}{[(1-r^2)^2 + 4\zeta^2 r^2]^{1/2}}$

$|X| = 0.00483 = 4.83$ mm

Both types of damping give similar amplitudes at $r = 1.2768$ because $|D| \approx 1/(1-r^2)$ away from $r = 1 \quad \Leftarrow$

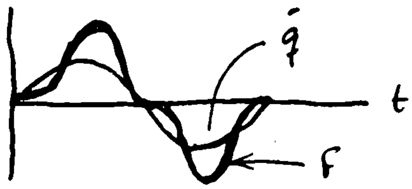
Exercise 3.17



$$f = \beta|\dot q|^2 \operatorname{sgn}(\dot q) = \beta|\dot q|^2\left(\frac{\dot q}{|\dot q|}\right) = \beta|\dot q|\dot q$$

Define delayed time: $t' = t - \phi/\omega$

Then $\quad q = X\cos(\omega t') \Rightarrow \dot q = X\omega\sin(\omega t')$



$$E_{dis} = \int_0^{2\pi/\omega} f\dot q\, dt$$

$$= 2\int_0^{\pi/\omega} f\dot q\, dt$$

$$= 2\beta X^3\omega^3 \int_0^{\pi/\omega} \sin(\omega t')^3\, dt'$$

$$\Rightarrow \qquad E_{dis} = \frac{8}{3}\beta\omega^2 X^3$$

$$\Rightarrow \text{Set } E_{dis} = \pi\omega C_{eq} X^2 \Rightarrow C_{eq} = \frac{8}{3\pi}\beta\omega X$$

$$\Rightarrow \quad \gamma_{eq} = \frac{\alpha}{\pi k} = \frac{\omega C_{eq}}{K} = \frac{8}{3\pi}\frac{\beta\omega^2}{K}X$$

From eq (3.2.25)

$$X = \frac{F}{k}\frac{1}{(1 - r^2 + i\gamma_{eq})}$$

Set $k = M\omega_{nat}^2 \Rightarrow \gamma_{eq} = \frac{8}{3\pi}\frac{\beta}{M}r^2 X$

$$\Rightarrow \quad \frac{kX}{F}\left[1 - r^2 + i\left(\frac{8}{3\pi}\frac{\beta}{M}r^2 X\right)\right] = 1$$

; quadratic eq for X
Complex root is OK

Exercise 3.14

$(\mathcal{P}_{dis})_{av} = 1000$ watt over $100 < \frac{\omega}{2\pi} < 1000$ Hz $\Rightarrow |X| = 0.008$ m

$E_{dis} = \mathcal{P}_{dis}(T) = \frac{2000\pi}{\omega} = \pi\omega C_{eq}|X|^2$

so $C_{eq} = \frac{2000}{\omega^2|X|^2}$

Then $C_{eq} = \frac{\alpha}{\pi\omega}$ & $\gamma_{eq} = \frac{\alpha}{\pi K} \Rightarrow \gamma_{eq} = \frac{\omega C_{eq}}{K} = \frac{2000}{\omega K|X|^2} = \frac{3.125(10^7)}{\omega K}$

$90°$ phase lag $\Rightarrow 200$ Hz $= \frac{\omega_{nat}}{2\pi} \Rightarrow \omega_{nat} = 400\pi$

Then $K = M\omega_{nat}^2 = 7.896(10^5)$ N/m

Thus $\gamma_{eq} = \frac{39.58}{\omega} = \frac{39.58}{r\omega_{nat}} = \frac{0.03150}{r}$ $\Leftarrow$

Set $|X| = \frac{|F|}{k}\frac{1}{[(1-r^2)^2+\gamma_{eq}^2]^{1/2}}$

so $|F| = k|X|[(1-r^2)^2+\gamma_{eq}^2]^{1/2}$

$= 7.896(10^5)(0.008)\left[(1-r^2)^2 + \frac{0.0315^2}{r^2}\right]^{1/2}$

$\omega = 100(2\pi) \Rightarrow r = 0.5 \Rightarrow |F| = 4754$ N

$\omega = 200(2\pi) \Rightarrow r = 1 \Rightarrow |F| = 198.94$ N

$\omega = 1000(2\pi) \Rightarrow r = 5 \Rightarrow |F| = 151.6$ kN

$\text{Max}(|F|) \Rightarrow \frac{d|F|}{dr} = 0$

Let $U = (1-r^2)^2 + \frac{0.0315^2}{r^2} \Rightarrow |F| = 6317U^{1/2}$

$\frac{d|F|}{dr} = 6317(\tfrac{1}{2})U^{-1/2}\frac{dU}{dr} = 0 \Rightarrow \frac{dU}{dr} = 0$

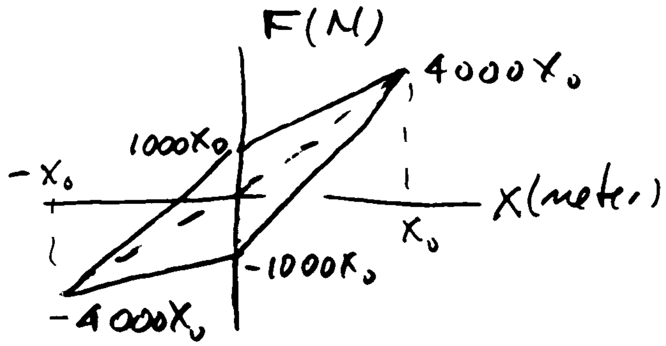
i.e. $2(1-r^2)(-2r) + 0.0315^2\left(-\frac{2}{r^3}\right) = 0$

But $100 \text{ Hz} < \omega < 1 \text{ kHz} \Leftrightarrow 0.5 < r < 5$

The last term is very small in this range, so

$\frac{d|F|}{dr} = 0 \Rightarrow r \approx 1 \Rightarrow \omega = \omega_{nat}$ $\Leftarrow$

Exercise 3.15



Given suspension force vs x indep of ω, $M = 10$ kg. Find (a) equivalent linear K & C, (b) X_{ss} when $F = 400 \sin(50t) + 300 \cos(50t)$ N

Solution: The diagonal line represents the mean value which is the contribution of the spring, so

$$K_{eq} = 4000 \text{ N/m} \quad \Leftarrow$$

The area enclosed by the curve is E_{dis} for a cycle:

$$E_{dis} = 2\left[\tfrac{1}{2}(2000 X_0)(X_0)\right] \quad \text{(two triangles)}$$

Equate $E_{dis} = \pi \omega C_{eq} X_0^2 \Rightarrow C_{eq} = \dfrac{2000.}{\pi\omega} \quad \Leftarrow$

Thus at $\omega = 50$ rad/s:

$$M\ddot{x} + C_{eq}\dot{x} + K_{eq}x = 400\sin(50t) + 300\cos(50t)$$

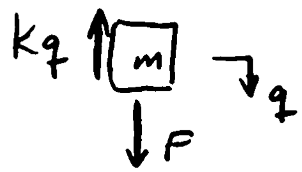
$$10\ddot{x} + 12.733\dot{x} + 4000x = Re\left[\left(\frac{400}{i} + 300\right)e^{i50t}\right]$$

Set $x = Re(Xe^{i50t})$

$$\left[-10(50^2) + 12.733(50i) + 4000\right]X = \left(\frac{400}{i} + 300\right)$$

$$X = -0.01485 + 0.01860i \text{ meter} \quad \Leftarrow$$

Exercise 3.16



When $q = Re[Y \exp(i\omega t)]$ then

$$M\ddot{q} + K(\omega) q = F = Re(F_0 \exp(i\omega t))$$

give $[K(\omega) - \omega^2 M] Y = F_0$

$$Y = \frac{F_0}{K(\omega) - \omega^2 M}$$

Static deflection ($\omega = 0$) $\quad \delta = \dfrac{Mg}{K(0)}$

Rewrite $Y = \dfrac{F_0}{K(0)} \dfrac{1}{\left[\dfrac{K(\omega)}{K(0)} - \omega^2 \dfrac{M}{K(0)}\right]}$

but $\dfrac{M}{K(0)} = \dfrac{\delta}{g} \equiv \dfrac{1}{\omega_{nat}^2}$

Define $r = \dfrac{\omega}{\omega_{nat}}$, $\alpha = \dfrac{K(\omega)}{K(0)}$

so $\quad Y = \dfrac{F_0}{mg} \delta \dfrac{1}{\alpha - r^2}$

To evaluate, note that $E(0) = E_0$, so

$$\alpha = 1 + \left(\frac{E_\infty}{E_0} - 1\right) \frac{\omega\tau}{\omega^2\tau^2 + 1} (\omega\tau + i)$$

$$= 1 + \left(\frac{E_\infty}{E_0} - 1\right) \frac{r\,\omega_{nat}\tau}{r^2(\omega_{nat}\tau)^2 + 1} \left[r(\omega_{nat}\tau) + i\right]$$

Plot $|Y|$ vs r for $0.75 \leq r < 1.25$ with $\omega_{nat}\tau$ fixed

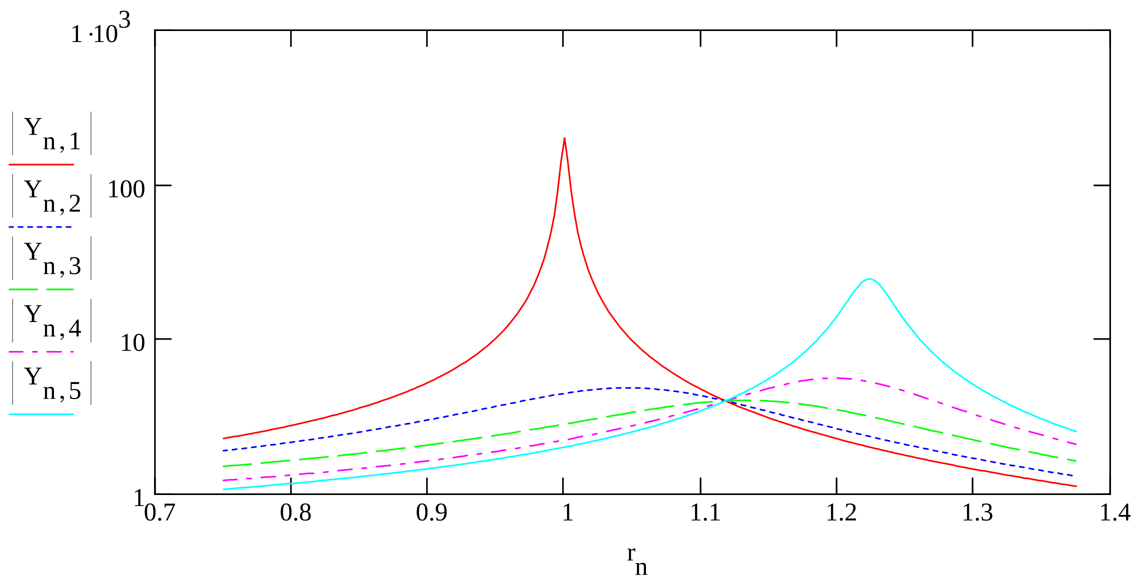
$$\alpha\left(r,\tau\omega_{nat}\right) := 1 + (1.5 - 1)\cdot\frac{r\cdot\tau\omega_{nat}}{\left(r\cdot\tau\omega_{nat}\right)^2 + 1}\cdot\left(r\cdot\tau\omega_{nat} + i\right)$$

$N := 251$ $\quad n := 1..N$ $\quad r_n := 0.75 + (n-1)\cdot\frac{0.50}{200}$

$\tau\omega_{nat_1} := 0.01$ $\quad \tau\omega_{nat_2} := 0.5$ $\quad \tau\omega_{nat_3} := 1.0$ $\quad \tau\omega_{nat_4} := 2.0$ $\quad \tau\omega_{nat_5} := 10.0$

$j := 1..5$

$$Y_{n,j} := \frac{1}{\left[\alpha\left(r_n,\tau\omega_{nat_j}\right) - \left(r_n\right)^2\right]}$$



Third case, $\omega_{nat}\tau = 1$, gives the smallest peak value of |Y|

Quality factors:

Frequency for peaks: $\omega_{peak_1} := 1$ $\quad \omega_{peak_2} := 1.05$ $\quad \omega_{peak_3} := 1.14$

$\omega_{peak_4} := 1.21$ $\quad \omega_{peak_5} := 1.23$

$$Y_{peak_j} := \max\left(\overrightarrow{\left|Y^{\langle j\rangle}\right|}\right)$$

Find half-power points by inspection:

$0.71 \cdot Y_{peak_1} = 142.007$

$\overrightarrow{\left|Y^{\langle 1 \rangle}\right|}^{T} =$

	95	96	97	98	99	100	101	102	103	104
1	33.1	39.4	48.7	63.3	89.4	141	200	141.9	89.5	63.2

$\Delta\omega_1 := r_{102} - r_{100}$

$0.71 \cdot Y_{peak_2} = 3.451$

$\overrightarrow{\left|Y^{\langle 2 \rangle}\right|}^{T} =$

	70	71	72	73	74	75	76	77	78	79
0	3.327	3.361	3.396	3.431	3.467	3.503	3.539	3.576	3.613	3.651

$\Delta\omega_2 := r_{159} - r_{74}$

$0.71 \cdot Y_{peak_3} = 2.862$

$\overrightarrow{\left|Y^{\langle 3 \rangle}\right|}^{T} =$

	96	97	98	99	100	101	102	103	104	105
1	2.709	2.732	2.756	2.78	2.804	2.828	2.853	2.879	2.904	2.93

$\Delta\omega_3 := r_{199} - r_{101}$

$0.71 \cdot Y_{peak_4} = 3.99$

$\overrightarrow{\left|Y^{\langle 4 \rangle}\right|}^{T} =$

	204	205	206	207	208	209	210	211	212	213
1	4.392	4.319	4.247	4.175	4.104	4.034	3.965	3.897	3.83	3.764

$\Delta\omega_4 := r_{210} - r_{148}$

$0.71 \cdot Y_{peak_5} = 17.451$

$\overrightarrow{\left|Y^{\langle 5 \rangle}\right|}^{T} =$

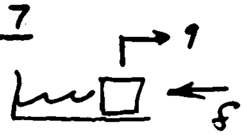
	180	181	182	183	184	185	186	187	188	189
0	14.27	15.32	16.48	17.74	19.1	20.51	21.89	23.13	24.07	24.58

$\Delta\omega_5 := r_{183} - r_{149}$

$$QF_j := \frac{\omega_{peak_j}}{\Delta\omega_j}$$

$$QF^T = (200 \quad 4.941 \quad 4.653 \quad 7.806 \quad 14.471)$$

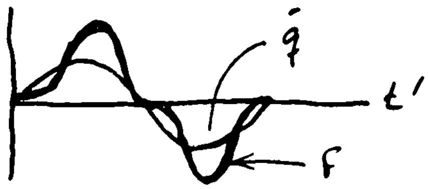
Exercise 3.17



$$f = \beta|\dot q|^2 \operatorname{sgn}(\dot q) = \beta|\dot q|^2\left(\frac{\dot q}{|\dot q|}\right) = \beta|\dot q|\,\dot q$$

Define delayed time: $t' = t - \phi/\omega$

Then $q = X\cos(\omega t') \Rightarrow \dot q = X\omega\sin(\omega t')$



$$E_{dis} = \int_0^{2\pi/\omega} f\dot q\,dt$$

$$= 2\int_0^{\pi/\omega} f\dot q\,dt$$

$$= 2\beta X^3\omega^3\int_0^{\pi/\omega} \sin(\omega t')^3\,dt'$$

$$E_{dis} = \frac{8}{3}\beta\omega^2 X^3 \qquad \Leftarrow$$

Set $E_{dis} = \pi\omega C_{eq}X^2 \Rightarrow C_{eq} = \frac{8}{3\pi}\beta\omega X \qquad \Leftarrow$

$$\gamma_{eq} = \frac{\alpha}{\pi k} = \frac{\omega C_{eq}}{K} = \frac{8}{3\pi}\frac{\beta\omega^2}{K}X \qquad \Leftarrow$$

From eq (3.2.25)

$$X = \frac{F}{k}\frac{1}{(1 - r^2 + i\gamma_{eq})}$$

Set $k = M\omega_{nat}^2 \Rightarrow \gamma_{eq} = \frac{8}{3\pi}\frac{\beta}{M}r^2X$

$$\frac{kX}{F}\left[1 - r^2 + i\left(\frac{8}{3\pi}\frac{\beta}{M}r^2X\right)\right] = 1 \qquad \Leftarrow$$

quadratic eq for X, Complex root is OK

Exercise 3.18

$$M\ddot{q} + Kq = F\cos(\omega_{nat} t), \quad q(0) = \dot{q}(0) = 0$$

$M = 100$ kg & $\frac{Mg}{k} = 0.005$ meter $\Rightarrow k = \frac{100(9.807)}{0.005}$

$\omega_{nat} = \left(\frac{k}{M}\right)^{1/2} = 44.29$ rad/s

$\frac{F}{k} = 0.0004$ meter $\Rightarrow F = 78.46$ N

To use eq (3.3.4), $Q(t) = F\cos(\omega_{nat} t) \Rightarrow \psi = 0$

Thus $q = \frac{F}{2M\omega_{nat}} t \sin(\omega_{nat} t) + C_1 \cos(\omega_{nat} t)$
$+ C_2 \sin(\omega_{nat} t)$

Satisfy i.c. $q(0) = C_1 = 0$

$\dot{q}(0) = \omega_{nat} C_2 = 0$

Thus $q = 0.008858 t \sin(44.29 t)$ meter $\Leftarrow$

A conservative estimate for $|q| < 0.020$ meters is based on the envelope function:

$0.008858\, t < 0.02$

$t < 2.258$ sec $\Leftarrow$

If the force were $F\sin(\omega_{nat} t)$, the particular solution would be

$$q_p = -\frac{F}{2M\omega_{nat}} t \cos(\omega_{nat} t)$$

In that case q_c would not be zero, so setting $|q_p + q_c| < 0.02$ would lead to a different result