For moving interfaces with uniform surface tension separating Newtonian fluids, the tangential stress is matched on either side of the interface:

$$\eta_1 \left(\frac{\partial u_{\mathrm{t},1}}{\partial n} + \frac{\partial u_{\mathrm{n},1}}{\partial t} \right) = \eta_2 \left(\frac{\partial u_{\mathrm{t},2}}{\partial n} + \frac{\partial u_{\mathrm{n},2}}{\partial t} \right) . \tag{1.77}$$

1.11 Supplementary reading

Modern introductory texts that cover the basic fluid mechanical equations include Fox, Pritchard, and McDonald [17], Munson, Young, and Okiishi [18], White [19], and Bird, Stewart, and Lightfoot [20]. These texts progress through this material more methodically, and are a good resource for those with minimal fluids training. More advanced treatment can be found in Panton [21], White [22], Kundu and Cohen [23], or Batchelor [24]. Batchelor provides a particularly lucid description of the Newtonian approximation, the fundamental meaning of pressure in this context, and why its form follows naturally from basic assumptions about isotropy of the fluid. Texts on kinetic theory [25, 26] provide a molecular-level description of the foundations of the viscosity and Newtonian model.

Although the classical fluids texts are excellent sources for the governing equations, kinematic relations, constitutive relations, and classical boundary conditions, they typically do not treat slip phenomena at liquid–solid interfaces. An excellent and comprehensive review of slip phenomena at liquid–solid interfaces can be found in [27] and the references therein. Slip in gas–solid systems is discussed in [3].

The treatment of surface tension in this chapter is similar to that found in basic fluids texts [21, 24] but omits many critical topics, including surfactants. Reference [28] covers these topics in great detail and is an invaluable resource. References [29, 30] cover flows owing to surface tension gradients, e.g., thermocapillary flows. A detailed discussion of boundary conditions is found in [31].

Although porous media and gels are commonly used in microdevices, especially for chemical separations, this text focuses on bulk fluid flow in micro- and nanochannels and omits consideration of flow through porous media and gels. Reference [29] provides one source to describe these flows. Another fascinating rheological topic that is largely omitted here is the flow of particulate suspensions and granular systems, with blood being a prominent example. Discussions of biorheology can be found in [23], and colloid science texts [29, 32] treat particulate suspensions and their rheology.

1.12 Exercises

1.1 In general, the sum of the extensional strains ($\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$) in an incompressible system always has the same value. What is this value? Why is this value known? **Ans:** The sum of extensional strains is zero, because the sum of extensional strains is proportional to the local change in volume of the flow. Thus for incompressible flow, cons of mass indicates that the sum of these strains is zero.

1.2 For a 2D flow (no z velocity components and all derivatives with respect to z are zero), write the components of the strain rate tensor $\vec{\vec{\varepsilon}}$ in terms of velocity derivatives.

Ans:
$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

1.3 Given the following strain rate tensors, draw a square-shaped fluid element and then show the shape that fluid element would take after being deformed by the fluid flow.

(a) $$\vec{\vec{\varepsilon}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) $$\vec{\vec{\varepsilon}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) $$\vec{\vec{\varepsilon}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Ans: See Fig. 1.24.

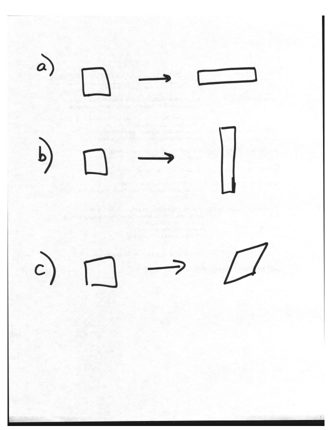

Figure 1.24: Fluid deformation owing to strain rate tensor.

1.4 The following strain rate tensor is not valid for incompressible flow. Why?

$$\vec{\vec{\varepsilon}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{1.78}$$

Ans: The sum of the extensional strains is nonzero; thus this violates conservation of mass

1.5 Could the following tensor be a strain rate tensor? If yes, explain the two properties that this tensor satisfies that make it valid. If no, explain why this tensor could not be a strain rate tensor.

$$\vec{\vec{\varepsilon}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix}. \tag{1.79}$$

Ans: No. This tensor is not symmetric.

1.6 Consider an incompressible flow field in cylindrical coordinates with axial symmetry (for example, a laminar jet issuing from a circular orifice). The axial symmetry implies that the flow field is a function of r and z but not θ. Can a stream function be derived for this case? If so, what is the relation between the derivatives of the stream function and the r and z velocities?

Solution: Yes. conservation of mass in axisymmetric coordinates is:

$$\nabla \cdot \vec{u} = 0 \tag{1.80}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r u_r + \frac{\partial}{\partial z} u_z = 0 \tag{1.81}$$

now we need to define a stream function ψ such that, if u_r and u_z are defined in terms of this stream function, conservation of mass is satisfied automatically. If we define ψ such that

$$\frac{\partial \psi}{\partial r} = u_z \tag{1.82}$$

and

$$\frac{\partial \psi}{\partial z} = -u_r\,, \tag{1.83}$$

which is similar to what we use for Cartesian coordinates, this will not work. There is still an r inside the $\frac{\partial}{\partial r}$ derivative for the radial term.

Instead, try defining ψ such that

$$\frac{\partial \psi}{\partial z} = r u_r\,. \tag{1.84}$$

Substituting this in, find

$$\frac{1}{r}\frac{\partial}{\partial r}\frac{\partial \psi}{\partial z} + \frac{\partial}{\partial z} u_z = 0\,. \tag{1.85}$$

Rearrange to get

$$\frac{\partial}{\partial z} u_z = -\frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial z}\psi \tag{1.86}$$

and then

$$\frac{\partial}{\partial z} u_z = \frac{\partial}{\partial z}\left(-\frac{1}{r}\frac{\partial}{\partial r}\psi\right)\,. \tag{1.87}$$

From this, we obtain

$$\frac{\partial \psi}{\partial r} = -r u_z\,, \tag{1.88}$$

so we find that

$$\frac{\partial \psi}{\partial z} = r u_r \tag{1.89}$$

$$\frac{\partial \psi}{\partial r} = -r u_z \tag{1.90}$$

satisfies the stream function requirements. Also, both of these relations could be multiplied by any constant, and conservation of mass would still be satisfied.

1.7 Consider the following two velocity gradient tensors:

(a) $\nabla \vec{\boldsymbol{u}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) $\nabla \vec{\boldsymbol{u}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Draw the streamlines for each velocity gradient tensor. With respect to the coordinate axes, identify which of these exhibits extensional strain and which exhibits shear strain. Following this, redraw the streamlines for each on axes that have been rotated 45° counterclockwise, using $x' = x/\sqrt{2} + y/\sqrt{2}$ and $y' = -x/\sqrt{2} + y/\sqrt{2}$. Are your conclusions about extensional and shear strain the same for the flow once you have rotated the axes? Do the definitions of extensional and shear strain depend on the coordinate system?

Solution:

See Fig. 1.25. The first exhibits shear strain and the second extensional strain with respect to the x and y axes. If we redraw with respect to rotated axes, the first exhibits extensional strain and the second shear strain. For isotropic materials, extensional and shear strain are different only in terms of how the shear is aligned with respect to an arbitrary coordinate system, and thus this distinction is relatively unimportant. For anisotropic materials (complex fluids or crystalline solids), the relation of the shear to the orientation of the material *is* important.

1.8 Write out the components of $\rho\vec{\boldsymbol{u}} \cdot \nabla\vec{\boldsymbol{u}}$ by using Cartesian derivatives. Is $\rho\vec{\boldsymbol{u}} \cdot \nabla\vec{\boldsymbol{u}}$ a scalar, vector, or second-order tensor?

Solution:

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the solution for this problem is not available
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1.9 Give an example of a 3D velocity gradient tensor that corresponds to a purely rotational flow.

Solution: A purely rotational flow has an antisymmetric velocity gradient tensor; thus any antisymmetric velocity gradient tensor answers this question.

1.10 Consider a differential Cartesian control volume and examine the convective momentum fluxes into and out of the control volume. Show that, for an incompressible fluid, the net convective outflow of momentum per unit volume is given by $\rho\vec{\boldsymbol{u}} \cdot \nabla\vec{\boldsymbol{u}}$.

Solution: Define a Cartesian control volume with sides of differential length dx, dy, and dz. Define the velocity at the center of the control volume equal to $\vec{\boldsymbol{u}}$=(u,v,w). Velocities at the faces are thus given by a first-order Taylor series expansion, which is exact for vanishingly small dx, dy, dz:

left face. At the left face, the velocity components are given by $u - \frac{1}{2}\frac{\partial u}{\partial x}dx$, $v - \frac{1}{2}\frac{\partial v}{\partial x}dx$, and $w - \frac{1}{2}\frac{\partial w}{\partial x}dx$.

right face. At the right face, the velocity components are given by $u + \frac{1}{2}\frac{\partial u}{\partial x}dx$, $v + \frac{1}{2}\frac{\partial v}{\partial x}dx$, and $w + \frac{1}{2}\frac{\partial w}{\partial x}dx$.

bottom face. At the bottom face, the velocity components are given by $u - \frac{1}{2}\frac{\partial u}{\partial y}dy$, $v - \frac{1}{2}\frac{\partial v}{\partial y}dy$, and $w - \frac{1}{2}\frac{\partial w}{\partial y}dy$.

top face. At the top face, the velocity components are given by $u + \frac{1}{2}\frac{\partial u}{\partial y}dy$, $v + \frac{1}{2}\frac{\partial v}{\partial y}dy$, and $w + \frac{1}{2}\frac{\partial w}{\partial y}dy$.

back face. At the back face, the velocity components are given by $u - \frac{1}{2}\frac{\partial u}{\partial z}dz$, $v - \frac{1}{2}\frac{\partial v}{\partial z}dz$, and $w - \frac{1}{2}\frac{\partial w}{\partial z}dz$.

front face. At the front face, the velocity components are given by $u + \frac{1}{2}\frac{\partial u}{\partial z}dz$, $v + \frac{1}{2}\frac{\partial v}{\partial z}dz$, and $w + \frac{1}{2}\frac{\partial w}{\partial z}dz$.

At any surface of a control volume, the outgoing flux density of momentum in a coordinate direction is given by the product of the density, the outward-pointing velocity *normal to the surface*, and the component of velocity *in that coordinate direction*. The velocity normal to the surface is given by $\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{n}}$, where $\vec{\boldsymbol{n}}$ is the unit outward normal.

For example, the momentum flux density of x momentum traveling through a surface normal to the z axis is given by ρuw, and the momentum flux density of y momentum traveling through a surface normal to the y axis is given by ρvv. In all cases, the values of u, v, and w are the values *at the surface*.

So, the outward-pointing velocity for the six faces are: left face: $-u$, right face: u, bottom face: $-v$, top face: v, back face: $-w$, front face: w.

The momentum fluxes are given by the product of momentum flux densities with the surface area. The areas for the six faces are: left and right faces: $dydz$, bottom and top faces: $dxdz$, back and front faces: $dxdy$.

Thus the momentum fluxes are given as follows. For the left face:

$$\begin{bmatrix} -\rho\left(u - \frac{1}{2}\frac{\partial u}{\partial x}dx\right)\left(u - \frac{1}{2}\frac{\partial u}{\partial x}dx\right)dydz \\ -\rho\left(u - \frac{1}{2}\frac{\partial u}{\partial x}dx\right)\left(v - \frac{1}{2}\frac{\partial v}{\partial x}dx\right)dydz \\ -\rho\left(u - \frac{1}{2}\frac{\partial u}{\partial x}dx\right)\left(w - \frac{1}{2}\frac{\partial w}{\partial x}dx\right)dydz \end{bmatrix} \tag{1.91}$$

For the right face:

$$\begin{bmatrix} \rho\left(u + \frac{1}{2}\frac{\partial u}{\partial x}dx\right)\left(u + \frac{1}{2}\frac{\partial u}{\partial x}dx\right)dydz \\ \rho\left(u + \frac{1}{2}\frac{\partial u}{\partial x}dx\right)\left(v + \frac{1}{2}\frac{\partial v}{\partial x}dx\right)dydz \\ \rho\left(u + \frac{1}{2}\frac{\partial u}{\partial x}dx\right)\left(w + \frac{1}{2}\frac{\partial w}{\partial x}dx\right)dydz \end{bmatrix} \tag{1.92}$$

For the bottom face:

$$\begin{bmatrix} -\rho\left(v - \frac{1}{2}\frac{\partial v}{\partial x}dx\right)\left(u - \frac{1}{2}\frac{\partial u}{\partial x}dx\right)dydz \\ -\rho\left(v - \frac{1}{2}\frac{\partial v}{\partial x}dx\right)\left(v - \frac{1}{2}\frac{\partial v}{\partial x}dx\right)dydz \\ -\rho\left(v - \frac{1}{2}\frac{\partial v}{\partial x}dx\right)\left(w - \frac{1}{2}\frac{\partial w}{\partial x}dx\right)dydz \end{bmatrix} \tag{1.93}$$

For the top face:

$$\begin{bmatrix} \rho\left(v+\frac{1}{2}\frac{\partial v}{\partial x}dx\right)\left(u+\frac{1}{2}\frac{\partial u}{\partial x}dx\right)dydz \\ \rho\left(v+\frac{1}{2}\frac{\partial v}{\partial x}dx\right)\left(v+\frac{1}{2}\frac{\partial v}{\partial x}dx\right)dydz \\ \rho\left(v+\frac{1}{2}\frac{\partial v}{\partial x}dx\right)\left(w+\frac{1}{2}\frac{\partial w}{\partial x}dx\right)dydz \end{bmatrix} \tag{1.94}$$

For the back face:

$$\begin{bmatrix} -\rho\left(w-\frac{1}{2}\frac{\partial w}{\partial x}dx\right)\left(u-\frac{1}{2}\frac{\partial u}{\partial x}dx\right)dydz \\ -\rho\left(w-\frac{1}{2}\frac{\partial w}{\partial x}dx\right)\left(v-\frac{1}{2}\frac{\partial v}{\partial x}dx\right)dydz \\ -\rho\left(w-\frac{1}{2}\frac{\partial w}{\partial x}dx\right)\left(w-\frac{1}{2}\frac{\partial w}{\partial x}dx\right)dydz \end{bmatrix} \tag{1.95}$$

For the front face:

$$\begin{bmatrix} \rho\left(w+\frac{1}{2}\frac{\partial w}{\partial x}dx\right)\left(u+\frac{1}{2}\frac{\partial u}{\partial x}dx\right)dydz \\ \rho\left(w+\frac{1}{2}\frac{\partial w}{\partial x}dx\right)\left(v+\frac{1}{2}\frac{\partial v}{\partial x}dx\right)dydz \\ \rho\left(w+\frac{1}{2}\frac{\partial w}{\partial x}dx\right)\left(w+\frac{1}{2}\frac{\partial w}{\partial x}dx\right)dydz \end{bmatrix} \tag{1.96}$$

The sum of these six sources of momentum flux gives the net outward convective momentum flux. We neglect all terms with the product of two differential lengths, as these are small compared with terms with only one differential length. The sum of the left and right face fluxes is:

$$\begin{bmatrix} \rho\left(u\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial x}\right)dxdydz \\ \rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)dxdydz \\ \rho\left(u\frac{\partial w}{\partial x}+w\frac{\partial u}{\partial x}\right)dxdydz \end{bmatrix} \tag{1.97}$$

For the bottom and top faces, the sum of the fluxes is:

$$\begin{bmatrix} \rho\left(v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}\right)dxdydz \\ \rho\left(v\frac{\partial v}{\partial y}+v\frac{\partial v}{\partial y}\right)dxdydz \\ \rho\left(v\frac{\partial w}{\partial y}+w\frac{\partial v}{\partial y}\right)dxdydz \end{bmatrix} \tag{1.98}$$

For the back and front faces, the sum of the fluxes is:

$$\begin{bmatrix} \rho\left(w\frac{\partial u}{\partial z}+u\frac{\partial w}{\partial z}\right)dxdydz \\ \rho\left(w\frac{\partial v}{\partial z}+v\frac{\partial w}{\partial z}\right)dxdydz \\ \rho\left(w\frac{\partial w}{\partial z}+w\frac{\partial w}{\partial z}\right)dxdydz \end{bmatrix} \tag{1.99}$$

Summing together, we get

$$\begin{bmatrix} \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+u\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial y}+u\frac{\partial w}{\partial z}\right)dxdydz \\ \rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+v\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}+v\frac{\partial w}{\partial z}\right)dxdydz \\ \rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+w\frac{\partial u}{\partial x}+w\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial z}\right)dxdydz \end{bmatrix} \tag{1.100}$$

The last three terms of each momentum flux component sum to zero, because $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$. Thus the convective momentum flux is given by

$$\begin{bmatrix} \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)dxdydz \\ \rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)dxdydz \\ \rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)dxdydz \end{bmatrix} \tag{1.101}$$

and the convective momentum flux per unit volume is

$$\begin{bmatrix} \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) \\ \rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) \\ \rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) \end{bmatrix} \tag{1.102}$$

This expression is equal to $\rho\vec{\boldsymbol{u}}\cdot\nabla\vec{\boldsymbol{u}}$.

1.11 Consider a differential Cartesian control volume and examine the viscous stresses on the control volume. Do not use a particular model for these viscous stresses; simply assume that $\vec{\vec{\boldsymbol{\tau}}}_{\text{visc}}$ is known. Show that the net outflow of momentum from these forces is given by $\nabla\cdot\vec{\vec{\boldsymbol{\tau}}}_{\text{visc}}$.

Solution:

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the solution for this problem is not available
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1.12 Show that $\nabla\cdot 2\eta\vec{\vec{\boldsymbol{\varepsilon}}}=\eta\nabla^2\vec{\boldsymbol{u}}$ if the viscosity is uniform and the fluid is incompressible.

Solution: start with $2\eta\vec{\vec{\boldsymbol{\varepsilon}}}$:

$$2\eta\vec{\vec{\boldsymbol{\varepsilon}}} \tag{1.103}$$

$$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot 2\eta\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{pmatrix} \tag{1.104}$$

$$\eta \begin{pmatrix} 2\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y}\frac{\partial v}{\partial x}\right) + \left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial}{\partial z}\frac{\partial w}{\partial x}\right) \\ \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 v}{\partial x^2}\right) + 2\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial}{\partial z}\frac{\partial w}{\partial y}\right) \\ \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial^2 w}{\partial x^2}\right) + \left(\frac{\partial}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial^2 w}{\partial z^2} \end{pmatrix} \tag{1.105}$$

from equality of mixed partials and incompressibility, can subtract away half of the terms:

$$\eta \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{pmatrix} \tag{1.106}$$

1.13 For a one dimensional flow given by $\vec{\boldsymbol{u}} = u(y)$, the strain rate magnitude is given by $\frac{1}{2}\frac{\partial u}{\partial y}$ and the vorticity magnitude is given by $\frac{\partial u}{\partial y}$. Are the strain rate and vorticity proportional to each other in general? If not, why are they proportional in this case?

Solution:

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the solution for this problem is not available
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1.14 Write out the Navier–Stokes equations in cylindrical coordinates (see Appendix D). Simplify these equations for the case of plane symmetry.

Solution:

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the solution for this problem is not available
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1.15 Write out the Navier–Stokes equations in cylindrical coordinates (see Appendix D). Simplify these equations for the case of axial symmetry.

Solution:

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the solution for this problem is not available
```

1.16 Write out the Navier–Stokes equations in spherical coordinates (see Appendix D). Simplify these equations for the case of axial symmetry.

Solution:

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the solution for this problem is not available
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1.17 For each of the following Cartesian velocity gradient tensors, (1) calculate the strain rate tensor, (2) calculate the rotation rate tensor, and (3) sketch the streamlines for the flow:

(a) $\nabla u = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

(b) $\nabla u = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

(c) $\nabla u = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

(d) $\nabla u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

Solution: For $\nabla u = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$$\vec{\vec{\boldsymbol{\varepsilon}}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.107}$$

and

$$\vec{\vec{\boldsymbol{\omega}}} = 0\,. \tag{1.108}$$

See Fig. 1.26.

For $\nabla u = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$$\vec{\vec{\boldsymbol{\varepsilon}}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.109}$$

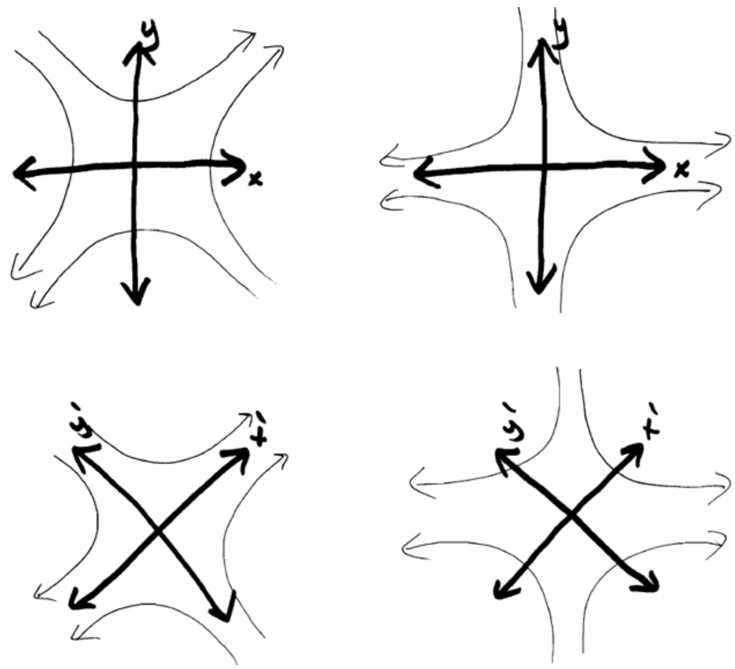

Figure 1.25: Streamlines for case 1 (left) and case 2 (right) drawn with respect to original (top) and rotated (bottom) axes.

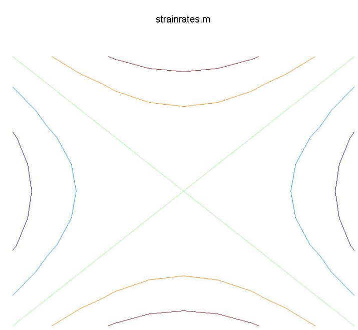

Figure 1.26: Flow streamlines.

and

$$\vec{\vec{\omega}} = 0\,. \tag{1.110}$$

See Fig. 1.27.

For $\nabla u = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$$\vec{\vec{\varepsilon}} = 0 \tag{1.111}$$

and

$$\vec{\vec{\omega}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{1.112}$$

See Fig. 1.28.

For $\nabla u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$$\vec{\vec{\varepsilon}} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.113}$$

and

$$\vec{\vec{\omega}} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{1.114}$$

See Fig. 1.29.

1.18 Consider the 2D flows defined by the following stream functions. The symbols A, B, C, and D denote constants.

(a) $\psi = Axy\,.$

Figure 1.27: Flow streamlines.

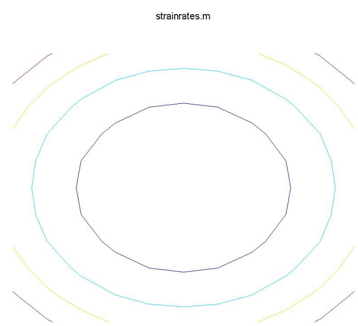

Figure 1.28: Flow streamlines.

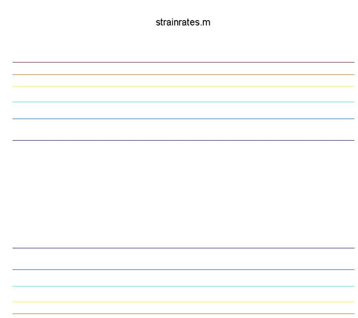

Figure 1.29: Flow streamlines.

(b) $\psi = \frac{1}{2}By^2$.

(c) $\psi = C \ln\left(\sqrt{x^2+y^2}\right)$.

(d) $\psi = -D\left(x^2+y^2\right)$.

For the flow field denoted by each of the preceding stream functions, execute the following:

(a) Show that the flow field satisfies conservation of mass.

(b) Derive the four components of the Cartesian strain rate tensor $\vec{\vec{\boldsymbol{\varepsilon}}}$.

(c) Plot streamlines for these flows in the regions $-5 < x < 5$ and $-5 < y < 5$.

(d) Assume that the pressure field $p(x,y)$ is known for each flow. Derive the four components of the Cartesian stress tensor $\vec{\vec{\boldsymbol{\tau}}}$, assuming that the fluid is Newtonian.

(e) Imagine that a 5×5 grid of lines (see Fig. 1.30) is visualized by instantaneously making a grid of tiny bubbles in a flow of water. Sketch the result if the grid were convected in the specified flow field starting at time $t = 0$ and a picture of the deformed grid was taken at a later time.

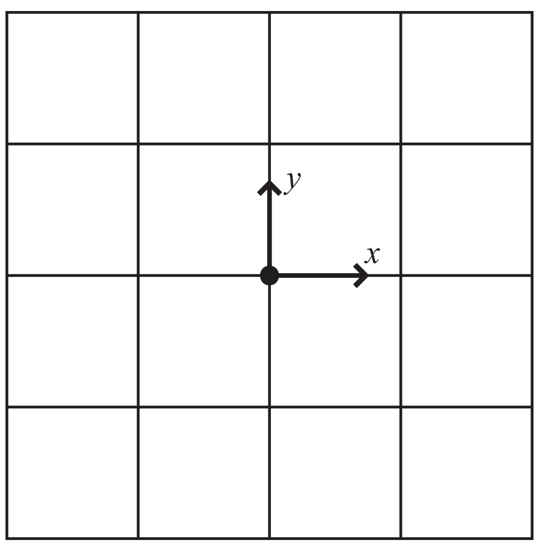

Figure 1.30: A grid that can be used to visualize how a flow deforms.

Solution:

conservation of mass All should obey conservation of mass, that is the point of a stream function.

$$\nabla \cdot \vec{\boldsymbol{u}} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\psi - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\psi = 0 \tag{1.115}$$

Cartesian strain rate tensor Strain rate tensor is defined in 2D Cartesian coordinates as:

$$\vec{\vec{\boldsymbol{\varepsilon}}} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} \end{pmatrix} \tag{1.116}$$

In terms of stream func:

$$\vec{\vec{\boldsymbol{\varepsilon}}} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial y}\psi & \frac{1}{2}\left(\frac{\partial^2}{\partial y^2}\psi - \frac{\partial^2}{\partial x^2}\psi\right) \\ \frac{1}{2}\left(\frac{\partial^2}{\partial y^2}\psi - \frac{\partial^2}{\partial x^2}\psi\right) & -\frac{\partial}{\partial x}\frac{\partial}{\partial y}\psi \end{pmatrix} \tag{1.117}$$

For $\psi = Axy$:

$$\vec{\vec{\boldsymbol{\varepsilon}}} = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \tag{1.118}$$

For $\psi = \frac{1}{2}By^2$:

$$\vec{\vec{\boldsymbol{\varepsilon}}} = \begin{pmatrix} 0 & \frac{1}{2}B \\ \frac{1}{2}B & 0 \end{pmatrix} \tag{1.119}$$

For $\psi = C\ln\left(\sqrt{x^2+y^2}\right)$:

$$\vec{\vec{\boldsymbol{\varepsilon}}} = \begin{pmatrix} \frac{-2Cxy}{(x^2+y^2)^2} & \frac{C\left(x^2-y^2\right)}{(x^2+y^2)^2} \\ \frac{C\left(x^2-y^2\right)}{(x^2+y^2)^2} & \frac{2Cxy}{(x^2+y^2)^2} \end{pmatrix} \tag{1.120}$$

For $\psi = -D\left(x^2+y^2\right)$:

$$\vec{\vec{\boldsymbol{\varepsilon}}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.121}$$

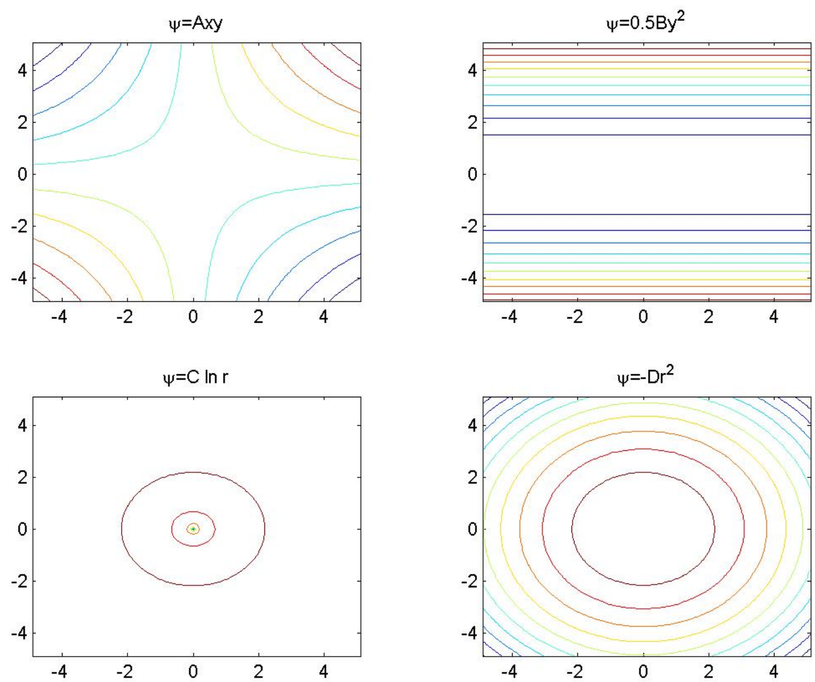

Figure 1.31: Streamlines for some simple flows. Source: simpleflows.m.

streamlines. see Fig. 1.31.

stress tensor. Stress tensor is given by:

$$\vec{\vec{\boldsymbol{\tau}}} = 2\eta\vec{\vec{\boldsymbol{\varepsilon}}} - p\vec{\vec{\boldsymbol{\delta}}} \tag{1.122}$$

In terms of stream func:

$$\vec{\vec{\boldsymbol{\varepsilon}}} = \begin{pmatrix} 2\eta\frac{\partial}{\partial x}\frac{\partial}{\partial y}\psi & \eta\left(\frac{\partial^2}{\partial y^2}\psi - \frac{\partial^2}{\partial x^2}\psi\right) \\ \eta\left(\frac{\partial^2}{\partial y^2}\psi - \frac{\partial^2}{\partial x^2}\psi\right) & -2\eta\frac{\partial}{\partial x}\frac{\partial}{\partial y}\psi \end{pmatrix} \tag{1.123}$$

For $\psi = Axy$:

$$\vec{\vec{\boldsymbol{\tau}}} = \begin{pmatrix} 2\eta A - p & 0 \\ 0 & -2\eta A - p \end{pmatrix} \tag{1.124}$$

For $\psi = \frac{1}{2}By^2$:

$$\vec{\vec{\boldsymbol{\tau}}} = \begin{pmatrix} -p & \eta B \\ \eta B & -p \end{pmatrix} \tag{1.125}$$

For $\psi = C\ln\left(\sqrt{x^2+y^2}\right)$:

$$\vec{\vec{\boldsymbol{\tau}}} = \begin{pmatrix} \frac{-4\eta Cxy}{(x^2+y^2)^2} - p & \frac{2\eta C\left(x^2-y^2\right)}{(x^2+y^2)^2} \\ \frac{2\eta C\left(x^2-y^2\right)}{(x^2+y^2)^2} & \frac{4\eta Cxy}{(x^2+y^2)^2} - p \end{pmatrix} \tag{1.126}$$

For $\psi = -D\left(x^2+y^2\right)$:

$$\vec{\vec{\boldsymbol{\tau}}} = \begin{pmatrix} -p & 0 \\ 0 & -p \end{pmatrix} \tag{1.127}$$

grid deformation. See Figures 1.32 through 1.35.

1.19 Create an infinitesimal control volume in cylindrical coordinates with edge lengths $d\imath$, $\imath d\theta$, and dz. Use the integral equation for conservation of mass:

$$\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho dV = -\int_S (\rho\vec{\boldsymbol{u}})\cdot\hat{\boldsymbol{n}}\,dA\,, \tag{1.128}$$

where V is the volume of the control volume, to derive the incompressible continuity equation in cylindrical coordinates.

Figure 1.32: Deformation of a grid by a stagnation flow, $\psi = Axy$. In general, grid spacing will be closer near origin and larger far from origin; this is not terribly clear from the sketch.

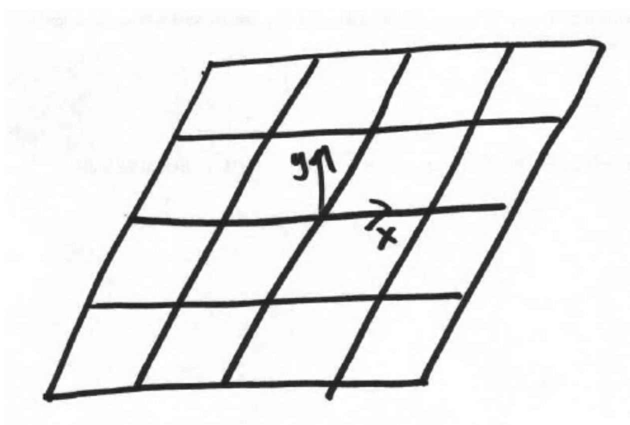

Figure 1.33: Deformation of a grid by a pure shear flow, $\psi = \frac{1}{2}By^2$.

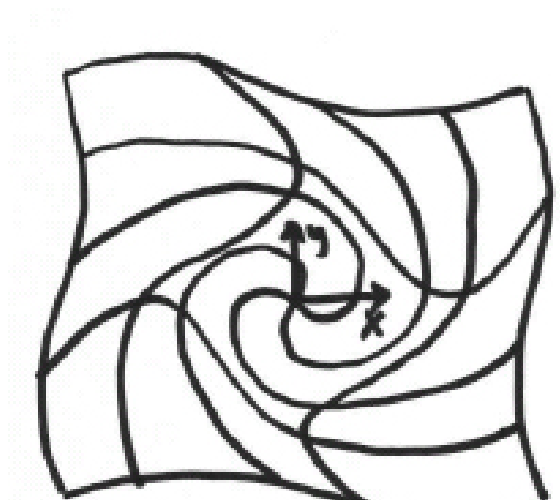

Figure 1.34: Deformation of a grid by an irrotational vortex, $\psi = C \ln\left(\sqrt{x^2+y^2}\right)$.

Solution: Evaluate $\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{n}}$ and the area on all six faces:

(a) bottom face: $\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{n}} = u_z$; $dS = r d\theta dr$
(b) top face: $\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{n}} = -u_z - \frac{\partial}{\partial z}u_z dz$; $dS = r d\theta dr$
(c) front face: $\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{n}} = u_\theta$; $dS = dr dz$
(d) back face: $\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{n}} = -u_\theta - \frac{\partial}{\partial \theta}u_\theta d\theta$; $dS = dr dz$
(e) left face: $\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{n}} = u_r$; $dS = r d\theta dz$
(f) right face: $\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{n}} = -u_r - \frac{\partial}{\partial r}u_r dr$; $dS = (r + dr)\, d\theta dz$

Plug these into the equation:

$$\frac{d}{dt}(\rho V) = 0 = \int_S \rho \vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{n}} dS \tag{1.129}$$

$$0 = \rho\left[r d\theta dr\left(u_z - u_z - \frac{\partial}{\partial z}u_z\right) + r d\theta dz u_r + (r + dr)d\theta dz\left(-u_r - \frac{\partial}{\partial r}u_r\right) + dr dz\left(u_\theta - u_\theta - \frac{\partial}{\partial \theta}u_\theta d\theta\right)\right] \tag{1.130}$$

divide by $\rho r dr d\theta dz$:

$$0 = -\frac{\partial}{\partial z}u_z - \frac{u_r}{r} - \frac{d}{dr}u_r - \frac{dr}{r}\frac{d}{dr}u_r - \frac{1}{r}\frac{\partial}{\partial \theta}u_\theta \tag{1.131}$$

Neglect terms proportional to dr/r, and divide by -1:

$$0 = \frac{\partial}{\partial z}u_z + \frac{u_r}{r} + \frac{d}{dr}u_r + \frac{1}{r}\frac{\partial}{\partial \theta}u_\theta \tag{1.132}$$

Note that $\frac{u_r}{r} + \frac{d}{dr}u_r$ is equal to $1/r$ times $\frac{d}{dr}(r u_r)$:

$$0 = \frac{1}{r}\frac{d}{dr}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}u_\theta + \frac{\partial}{\partial z}u_z \tag{1.133}$$

1.20 Using thermodynamic arguments, derive the Young–Laplace equation Eq. (1.41).

Solution:

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the solution for this problem is not available
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1.21 Use trigonometric and geometric arguments to deriveEq. (1.48).

Solution: By definition, the line from the center of curvature to the triple point is normal to the line tangent to the interface at that point. Thus $\alpha = \theta$ (see Fig. 1.36). From the triangle with (a) the triple point, (b) the center of curvature, and (c) the centerline of the capillary, we can observe that

$$\cos\theta = \frac{d/2}{R}\,, \tag{1.134}$$

and thus $R = d/2\cos\theta$.

1.22 Consider a capillary of diameter d oriented along the y axis and inserted into a reservoir of a fluid. Assume the surface tension of the liquid–gas interface is given by γ_{lg}. At the interface, the radius of curvature R can be assumed uniform everywhere in the xz plane (i.e., the interface is spherical) if the variations in the local pressure drop across the interface are small compared with the nominal value of the pressure drop across the interface.

(a) Write a relation for the pressure drop across the interface as a function of γ_{lg} and R.

(b) As a function of θ, evaluate the difference in height of the fluid at the center of the capillary with respect to the fluid at the outside edge of the capillary, and thus evaluate the difference in hydrostatic head between the center and edge of the capillary.

(c) The criterion for approximating the interface as spherical is that the pressure drop variations from center to edge are small compared with the pressure drop itself. Determine the maximum value for d for which the interface can be assumed spherical. Your result will be of the order of $\sqrt{\gamma_{\mathrm{lg}}/\rho g}$.

Solution:

pressure drop across interface The Young–Laplace equation gives

$$\Delta P = \frac{2\gamma_{\mathrm{lg}}}{R}\,. \tag{1.135}$$

height difference from center to edge At the center, the distance from the liquid column to the center of curvature of the interface is R. At the edge, the distance from the liquid column to the height of the center of curvature is $R\sin\theta$. Thus the difference in height is

$$\Delta h = R(1-\sin\theta)\,. \tag{1.136}$$

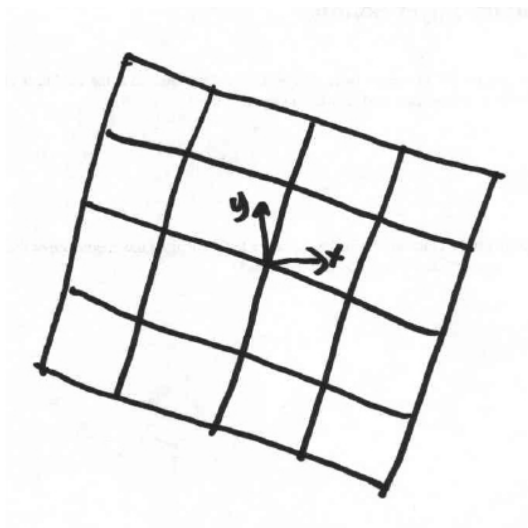

Figure 1.35: Deformation of a grid by solid body rotation, $\psi = -D\left(x^2+y^2\right)$.

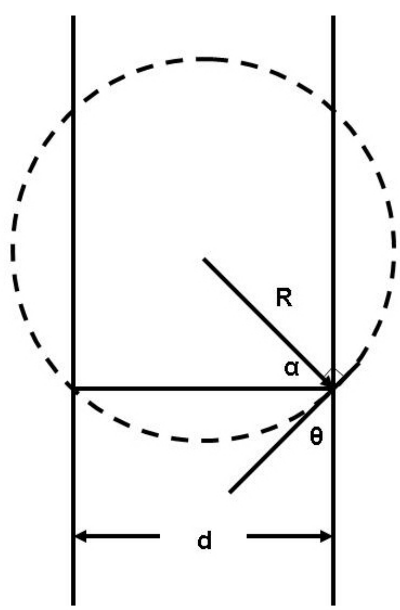

Figure 1.36: Geometry of interface in a capillary.

pressure difference from center to edge The difference between pressures at center and at edge is

$$\Delta P = \rho g R(1 - \sin\theta) \tag{1.137}$$

criterion for spherical interface shape The hydrostatic head difference between center and edge needs to be small compared with the interfacial pressure drop. Thus

$$\rho g R(1 - \sin\theta) < \frac{2\gamma_{\mathrm{lg}}}{R}\,. \tag{1.138}$$

Rearranging, we find

$$R^2 < \frac{1}{1-\sin\theta}\frac{2\gamma_{\mathrm{lg}}}{\rho g}\,. \tag{1.139}$$

Because $R = d/2\cos\theta$, we rearrange to find

$$d^2 < \frac{4\cos^2\theta}{1-\sin\theta}\frac{2\gamma_{\mathrm{lg}}}{\rho g}\,, \tag{1.140}$$

$$d < \sqrt{\frac{8\cos^2\theta}{1-\sin\theta}\frac{\gamma_{\mathrm{lg}}}{\rho g}}\,, \tag{1.141}$$

and finally

$$d < \sqrt{\frac{8(1-\sin^2\theta)}{1-\sin\theta}\frac{\gamma_{\mathrm{lg}}}{\rho g}}\,. \tag{1.142}$$

1.23 Show that the Euclidean norm of the rotation rate tensor is equal to $\sqrt{2}$ times the solid-body rotation rate of a point in a flow.

Solution:

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the solution for this problem is not available
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1.24 Draw a flat control volume at an interface between two domains and derive the general kinematic boundary condition for the normal velocities that is given in Eq. (1.54).

Solution:

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the solution for this problem is not available
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texts [22, 30] cover unsteady solutions by use of separation of variables. Asymptotic approximations used to describe perturbations to these flows are discussed in Bruus [33] and Leal [30], in addition to Van Dyke's classic monograph [34].

2.5 Exercises

2.1 Show that, if $\vec{\boldsymbol{u}} = (u,0,0)$, i.e., flow is in the x direction only, then $\vec{\boldsymbol{u}} \cdot \nabla \vec{\boldsymbol{u}} = (u\frac{\partial u}{\partial x},0,0)$.

Solution:

$$\vec{\boldsymbol{u}} = (u,v,w) = (u,0,0) \tag{2.42}$$

$$\nabla \vec{\boldsymbol{u}} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.43}$$

$$\vec{\boldsymbol{u}} \cdot \nabla \vec{\boldsymbol{u}} = (u\frac{\partial u}{\partial x},0,0) \tag{2.44}$$

2.2 Consider the following two cases:

(a) A Newtonian fluid.

(b) A power-law fluid, i.e., a fluid for which $\tau_{xy} = K\frac{du}{dy}\left|\frac{du}{dy}\right|^{n-1}$. You may simplify your math by assuming that $\frac{du}{dy}$ is positive.

Consider laminar flow between two infinite parallel plates, each aligned with the xz plane. The plates are located at $y = \pm h$. There are no applied pressure gradients. Assume the top plate moves in the x direction with velocity u_H and the bottom plate moves in the x direction with velocity u_L. For each case,

(a) Solve for the flow between the plates as a function of y.

(b) Derive relations for $u(y)$, $\vec{\vec{\boldsymbol{\tau}}}(y)$, and $\vec{\vec{\boldsymbol{\varepsilon}}}(y)$. Note that $\vec{\vec{\boldsymbol{\tau}}}$ and $\vec{\vec{\boldsymbol{\varepsilon}}}$ should both be second-order tensors. For each of these three parameters, comment on how the result is influenced by the magnitude of the viscosity as well as by its strain rate dependence.

(c) Evaluate the force per unit area that each surface must apply *to the fluid* to maintain this flow. Comment on how the result is influenced by the magnitude of the viscosity as well as by its strain rate dependence.