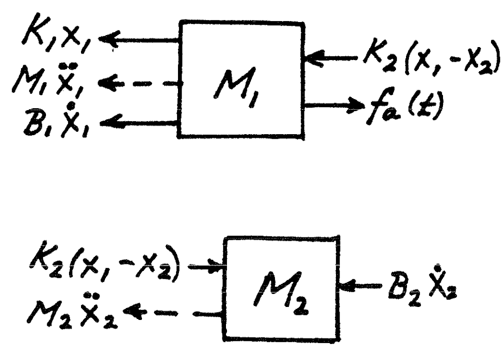
Chapter 2
Translational Mechanical Systems

Topic	Section	Examples	Problems[1]
Horizontal motion, applied force	4	1, 2	1*, 2, 3, 4*, 5, 6
Horizontal motion, displacement input	4	3	7*, 8, 9
Relative displacements	4	4	10*, 11, 12*, 13, 14
Vertical motion	4	5, 6	15*, 16, 17, 18*, 19, 20 21, 22*
Horizontal and vertical motion	4	7	23, 24
Series and parallel combinations	4	8, 9, 10, 11	25*, 26, 27*, 28, 29, 30

[1] The answers to problems with an asterisk are given in Appendix G.

2.1

Summing the forces shown on each of the free-body diagrams, we have



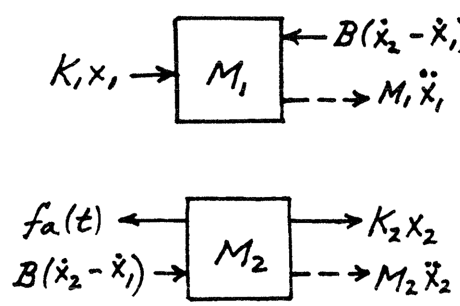
$$
\begin{aligned}
M_1\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 + K_2(x_1 - x_2) &= f_a(t) \\
M_2\ddot{x}_2 + B_2\dot{x}_2 - K_2(x_1 - x_2) &= 0
\end{aligned}
$$

Collecting terms gives

$$
\begin{aligned}
M_1\ddot{x}_1 + B_1\dot{x}_1 + (K_1 + K_2)x_1 - K_2x_2 &= f_a(t) \\
-K_2x_1 + M_2\ddot{x}_2 + B_2\dot{x}_2 + K_2x_2 &= 0
\end{aligned}
$$

2.2

Summing the forces shown on each of the free-body diagrams, we have



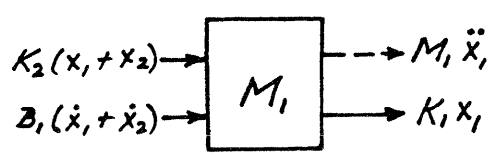
$$
\begin{aligned}
M_1\ddot{x}_1 + B(\dot{x}_1 - \dot{x}_2) + K_1x_1 &= 0 \\
M_2\ddot{x}_2 + B(\dot{x}_2 - \dot{x}_1) + K_2x_2 &= f_a(t)
\end{aligned}
$$

Collecting terms gives

$$
\begin{aligned}
M_1\ddot{x}_1 + B\dot{x}_1 + K_1x_1 - B\dot{x}_2 &= 0 \\
-B\dot{x}_1 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2 &= f_a(t)
\end{aligned}
$$

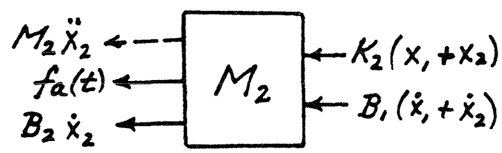
2.3

Summing the forces shown on each of the free-body diagrams, we have



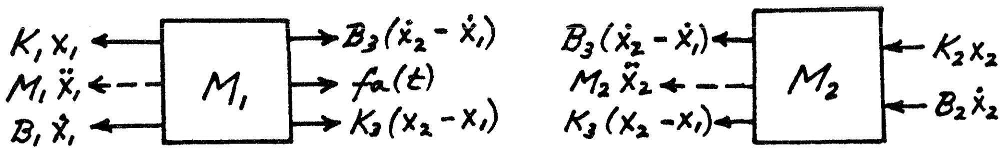
$$
\begin{aligned}
M_1\ddot{x}_1 + K_1x_1 + K_2(x_2 + x_1) + B_1(\dot{x}_2 + \dot{x}_1) &= 0 \\
M_2\ddot{x}_2 + B_2\dot{x}_2 + B_1(\dot{x}_2 + \dot{x}_1) + K_2(x_2 + x_1) + f_a(t) &= 0
\end{aligned}
$$

Collecting terms gives



$$\begin{aligned} M_1\ddot{x}_1 + B_1\dot{x}_1 + (K_1+K_2)x_1 + B_1\dot{x}_2 + K_2x_2 &= 0 \\ B_1\dot{x}_1 + K_2x_1 + M_2\ddot{x}_2 + (B_1+B_2)\dot{x}_2 + K_2x_2 &= -f_a(t) \end{aligned}$$

2.4

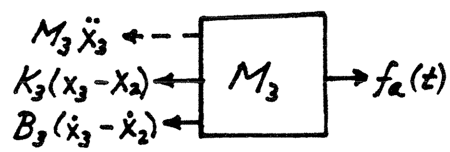


Summing the forces shown on each of the free-body diagrams and collecting terms, we get

$$\begin{aligned} M_1\ddot{x}_1 + (B_1+B_3)\dot{x}_1 + (K_1+K_3)x_1 - B_3\dot{x}_2 - K_3x_2 &= f_a(t) \\ -B_3\dot{x}_1 - K_3x_1 + M_2\ddot{x}_2 + (B_2+B_3)\dot{x}_2 + (K_2+K_3)x_2 &= 0 \end{aligned}$$

2.5

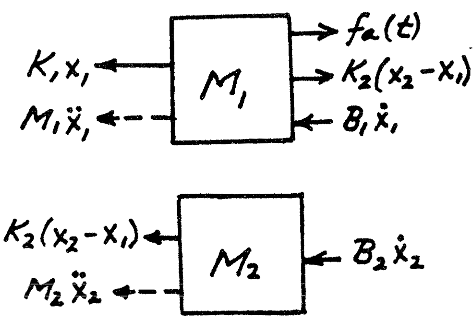
Summing the forces shown on each of the free-body diagrams and collecting terms, we get



$$\begin{aligned} M_1\ddot{x}_1 + (B_1+B_2)\dot{x}_1 + (K_1+K_2)x_1 - B_2\dot{x}_2 - K_2x_2 &= 0 \\ -B_2\dot{x}_1 - K_2x_1 + M_2\ddot{x}_2 + (B_2+B_3)\dot{x}_2 + (K_2+K_3)x_2 & \\ -B_3\dot{x}_3 - K_3x_3 &= 0 \\ -B_3\dot{x}_2 - K_3x_2 + M_3\ddot{x}_3 + B_3\dot{x}_3 + K_3x_3 &= f_a(t) \end{aligned}$$

2.6

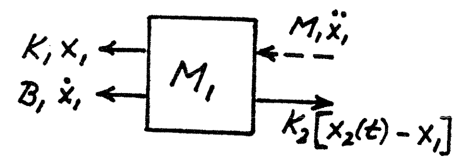
The free-body diagrams are drawn for the case where $x_3(t) = 0$. Summing the forces shown on each of the diagrams and collecting terms, we get



$$\begin{aligned} M_1\ddot{x}_1 + B_1\dot{x}_1 + (K_1+K_2)x_1 - K_2x_2 &= f_a(t) \\ -K_2x_1 + M_2\ddot{x}_2 + B_2\dot{x}_2 + K_2x_2 &= 0 \end{aligned}$$

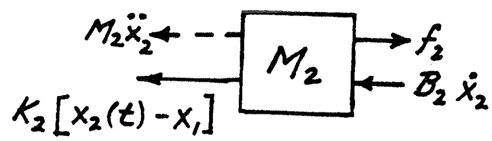
2.7

(a) Because the motion of M_2 is known, we need only the free-body diagram for M_1 in order to write the differential equation governing the motion of M_1. Summing the forces shown on that diagram and collecting terms, we get



$$M_1\ddot{x}_1 + B_1\dot{x}_1 + (K_1 + K_2)x_1 = K_2x_2(t)$$

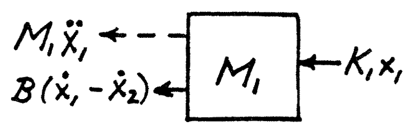
(b) Summing the forces shown on the free-body diagram for M_2 and rearranging the terms, we obtain



$$f_2 = -K_2x_1 + M_2\ddot{x}_2 + B_2\dot{x}_2 + K_2x_2(t)$$

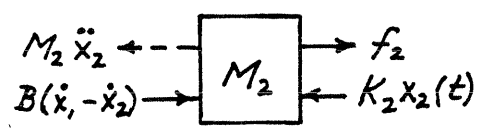
2.8

(a) Because the motion of M_2 is known, we need only the free-body diagram for M_1 in order to write the differential equation governing the motion of M_1. Summing the forces shown on that diagram and collecting terms, we get



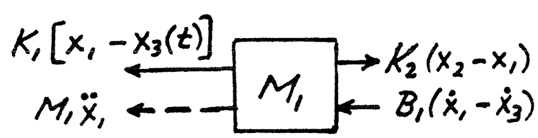
$$M_1\ddot{x}_1 + B\dot{x}_1 + K_1x_1 = B\dot{x}_2$$

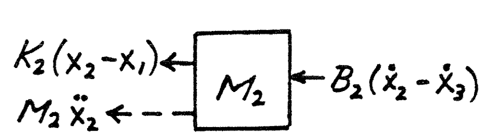
(b) Summing the forces shown on the free-body diagram for M_2 and rearranging the terms, we obtain



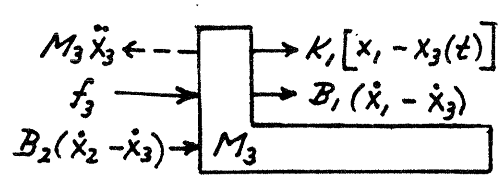
$$f_2 = -B\dot{x}_1 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2(t)$$

2.9





(a) Because the motion of M_3 is known, we need only the free-body diagrams for M_1 and M_2 in order to write the differential equations governing the motions of M_1 and M_2. Summing the forces shown on these diagrams and collecting terms, we get

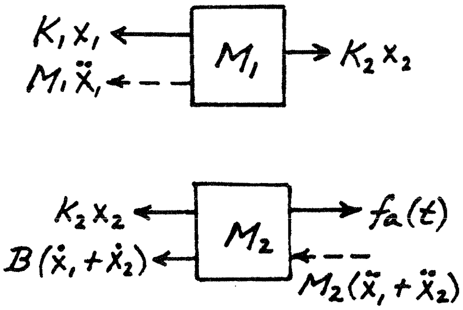


$$\begin{aligned} M_1\ddot{x}_1 + B_1\dot{x}_1 + (K_1 + K_2)x_1 - K_2x_2 &= B_1\dot{x}_3 + K_1x_3(t) \\ -K_2x_1 + M_2\ddot{x}_2 + B_2\dot{x}_2 + K_2x_2 &= B_2\dot{x}_3 \end{aligned}$$

(b) Summing the forces shown on the free-body diagram for M_3 and rearranging the terms, we obtain

$$f_3 = -B_1\dot{x}_1 - K_1x_1 - B_2\dot{x}_2 + M_3\ddot{x}_3 + (B_1 + B_2)\dot{x}_3 + K_1x_3(t)$$

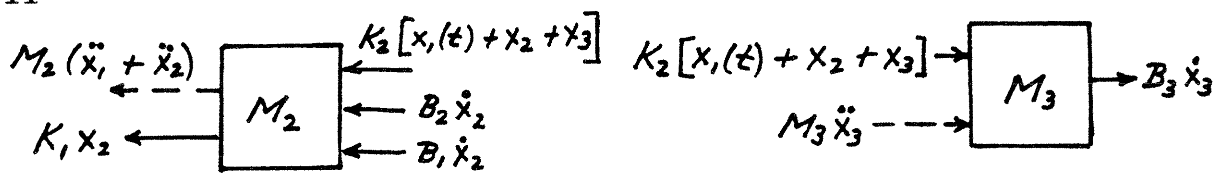
2.10



When drawing the free-body diagrams, note that x_2 is the relative displacement of M_2 with respect to M_1 and that the inertial force on M_2 depends on the absolute acceleration $\ddot{x}_1 + \ddot{x}_2$. Summing the forces shown on the diagrams and collecting terms, we have

$$\begin{aligned} M_1\ddot{x}_1 + K_1x_1 - K_2x_2 &= 0 \\ M_2\ddot{x}_1 + B\dot{x}_1 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2 &= f_a(t) \end{aligned}$$

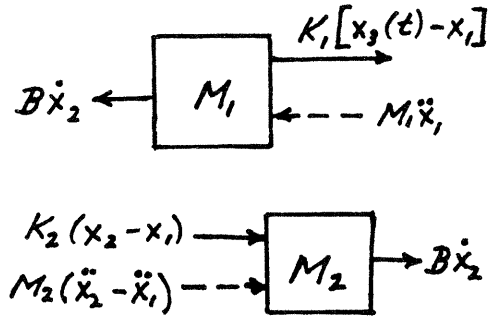
2.11



Because the motion of M_1 is known, we need only the free-body diagrams for M_2 and M_3. When drawing these diagrams, note that x_2 is the relative displacement of M_2 with respect to M_1 and that the absolute acceleration of M_2 is $\ddot{x}_1 + \ddot{x}_2$. Likewise, the compression of K_2 is $x_1(t) + x_2 + x_3$. Summing the forces shown on each of the diagrams and collecting terms, we get

$$\begin{aligned} M_2\ddot{x}_2 + (B_1 + B_2)\dot{x}_2 + (K_1 + K_2)x_2 + K_2x_3 &= -M_2\ddot{x}_1 - K_2x_1(t) \\ K_2x_2 + M_3\ddot{x}_3 + B_3\dot{x}_3 + K_2x_3 &= -K_2x_1(t) \end{aligned}$$

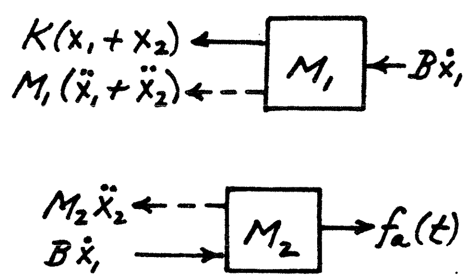
2.12



Because the motion of the right end of K_1 is known, we need only the free-body diagrams for M_1 and M_2. Because x_2 is a relative displacement, the compression of K_2 and the acceleration of M_2 are $x_2 - x_1$ and $\ddot{x}_2 - \ddot{x}_1$, respectively. Summing the forces shown on each of the diagrams and collecting terms, we get

$$\begin{aligned} M_1\ddot{x}_1 + K_1x_1 + B\dot{x}_2 &= K_1x_3(t) \\ -M_2\ddot{x}_1 - K_2x_1 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2 &= 0 \end{aligned}$$

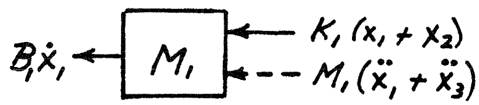
2.13

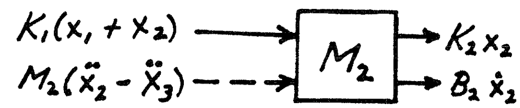


When drawing free-body diagrams for M_1 and M_2, note that x_1 is a relative displacement. The elongation of K is $x_1 + x_2$, and the absolute acceleration of M_1 is $\ddot{x}_1 + \ddot{x}_2$. Summing the forces shown on each of the diagrams and collecting terms, we get

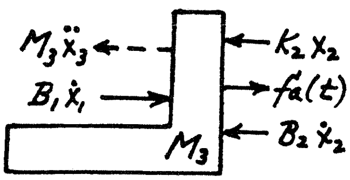
$$\begin{aligned} M_1\ddot{x}_1 + B\dot{x}_1 + Kx_1 + M_1\ddot{x}_2 + Kx_2 &= 0 \\ -B\dot{x}_1 + M_2\ddot{x}_2 &= f_a(t) \end{aligned}$$

2.14





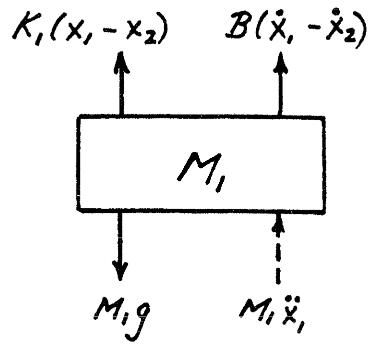
When drawing free-body diagrams, note that x_1 and x_2 are relative displacements. The elongation of K_1 is $x_1 + x_2$. The absolute accelerations of M_1 and M_2 are $\ddot{x}_1 + \ddot{x}_3$ and $\ddot{x}_2 - \ddot{x}_3$, respectively. Summing the forces shown on each of the diagrams and collecting terms, we get



$$
\begin{aligned}
M_1\ddot{x}_1 + B_1x_1 + K_1x_1 + K_1x_2 + M_1\ddot{x}_3 &= 0 \\
K_1x_1 + M_2\ddot{x}_2 + B_2\dot{x}_2 + (K_1+K_2)x_2 - M_2\ddot{x}_3 &= 0 \\
-B_1\dot{x}_1 + B_2\dot{x}_2 + K_2x_2 + M_3\ddot{x}_3 &= f_a(t)
\end{aligned}
$$

2.15

(a) Summing the forces shown on each of the free-body diagrams and collecting terms give



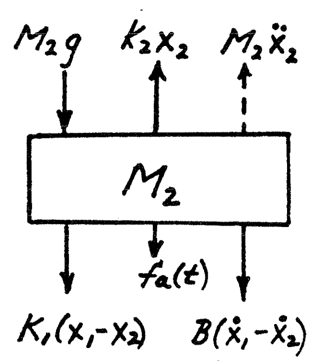
$$
\begin{aligned}
M_1\ddot{x}_1 + B\dot{x}_1 + K_1x_1 - B\dot{x}_2 - K_1x_2 &= M_1g \\
-B\dot{x}_1 - K_1x_1 + M_2\ddot{x}_2 + B\dot{x}_2 + (K_1+K_2)x_2 &= M_2g + f_a(t)
\end{aligned}
$$

(b) Letting $f_a(t) = 0$, replacing x_1 and x_2 by x_{1_0} and x_{2_0}, and noting that $\dot{x}_{1_0} = \ddot{x}_{1_0} = \dot{x}_{2_0} = \ddot{x}_{2_0} = 0$, we obtain

$$
\begin{aligned}
K_1x_{1_0} - K_1x_{2_0} &= M_1g \\
-K_1x_{1_0} + (K_1+K_2)x_{2_0} &= M_2g
\end{aligned}
$$

from which

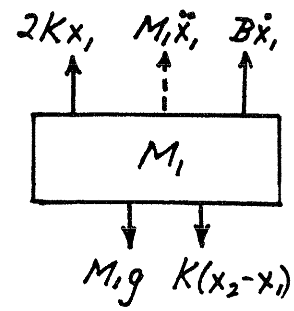
$$
x_{1_0} = \left(\frac{M_1+M_2}{K_2} + \frac{M_1}{K_1}\right)g \qquad \text{and} \qquad x_{2_0} = \left(\frac{M_1+M_2}{K_2}\right)g
$$



(c) We let $x_1 = x_{1_0} + z_1$ and $x_2 = x_{2_0} + z_2$, with x_{1_0} and x_{2_0} replaced by the expressions in part (b). We then substitute these expressions for x_1 and x_2 into the equations found in part (a). After canceling common terms, we get

$$
\begin{aligned}
M_1\ddot{z}_1 + B\dot{z}_1 + K_1z_1 - B\dot{z}_2 - K_1z_2 &= 0 \\
-B\dot{z}_1 - K_1z_1 + M_2\ddot{z}_2 + B\dot{z}_2 + (K_1+K_2)z_2 &= f_a(t)
\end{aligned}
$$

2.16



(a) From the free-body diagrams,

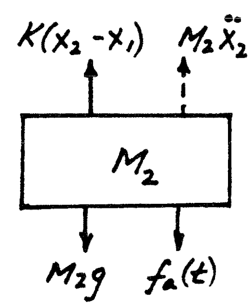
$$
\begin{aligned}
M_1\ddot{x}_1 + B\dot{x}_1 + 3Kx_1 - Kx_2 &= M_1 g \\
-Kx_1 + M_2\ddot{x}_2 + Kx_2 &= M_2 g + f_a(t)
\end{aligned}
$$

(b) With x_1 and x_2 replaced by the constant displacements x_{1_0} and x_{2_0}, and with $f_a(t) = 0$,

$$3x_{1_0} - x_{2_0} = \frac{M_1 g}{K} \quad \text{and} \quad -x_{1_0} + x_{2_0} = \frac{M_2 g}{K}$$

from which

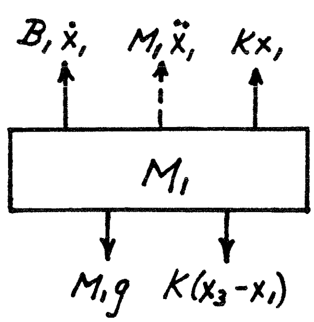
$$x_{1_0} = (M_1 + M_2)\frac{g}{2K} \quad \text{and} \quad x_{2_0} = (M_1 + 3M_2)\frac{g}{2K}$$



(c) Substituting $x_1 = x_{1_0} + z_1$ and $x_2 = x_{2_0} + z_2$ into the original differential equations, using the above expressions for x_{1_0} and x_{2_0}, and canceling common terms, we obtain

$$
\begin{aligned}
M_1\ddot{z}_1 + B\dot{z}_1 + 3Kz_1 - Kz_2 &= 0 \\
-Kz_1 + M_2\ddot{z}_2 + Kz_2 &= f_a(t)
\end{aligned}
$$

2.17



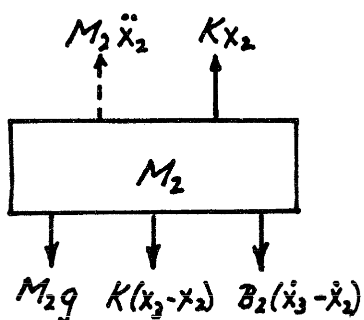
(a) Summing the forces shown on each of the free-body diagrams and collecting terms, we obtain

$$
\begin{aligned}
M_1\ddot{x}_1 + B_1\dot{x}_1 + 2Kx_1 - Kx_3 &= M_1 g \\
M_2\ddot{x}_2 + B_2\dot{x}_2 + 2Kx_2 - B_2\dot{x}_3 - Kx_3 &= M_2 g \\
-Kx_1 - B_2\dot{x}_2 - Kx_2 + M_3\ddot{x}_3 + B_2\dot{x}_3 + 2Kx_3 &= M_3 g + f_a(t)
\end{aligned}
$$

(b) Letting $f_a(t) = 0$, replacing x_1, x_2, and x_3 by the constant displacements x_{1_0}, x_{2_0}, and x_{3_0}, and noting that all the derivatives of these constant displacements are zero, we have the following three algebraic equations.

$$2x_{1_0} - x_{3_0} = \frac{M_1 g}{K}, \quad 2x_{2_0} - x_{3_0} = \frac{M_2 g}{K},$$

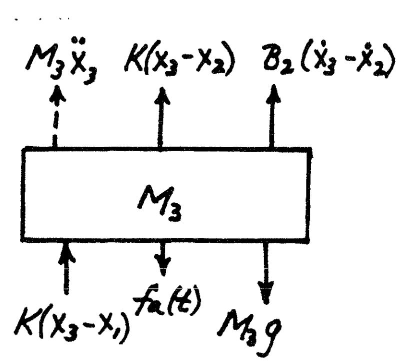
$$\text{and} \quad -x_{1_0} - x_{2_0} + 2x_{3_0} = \frac{M_3 g}{K}$$



Solving these equations simultaneously gives

$$
\begin{aligned}
x_{1_0} &= (3M_1 + M_2 + 2M_3)\frac{g}{4K} \\
x_{2_0} &= (M_1 + 3M_2 + 2M_3)\frac{g}{4K} \\
x_{3_0} &= (M_1 + M_2 + 2M_3)\frac{g}{2K}
\end{aligned}
$$

The four spring elongations are x_{1_0}, x_{2_0}, and

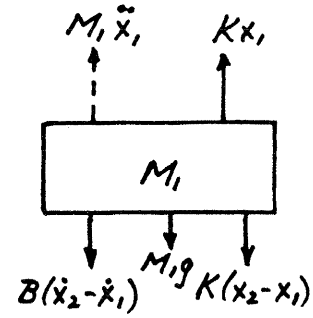


$$
\begin{aligned}
x_{3_0} - x_{1_0} &= (-M_1 + M_2 + 2M_3)\frac{g}{4K} \\
x_{3_0} - x_{2_0} &= (M_1 - M_2 + 2M_3)\frac{g}{4K}
\end{aligned}
$$

Note that the elongations are not affected by the vicous damping coefficients B_1 and B_2.

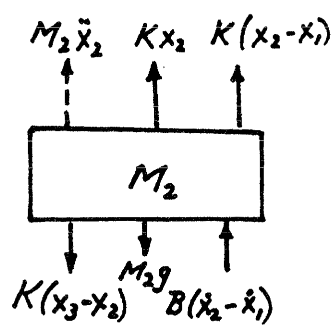
2.18

(a) Summing the forces shown on each of the free-body diagrams and collecting terms, we get



$$
\begin{aligned}
M_1\ddot{x}_1 + B\dot{x}_1 + 2Kx_1 - B\dot{x}_2 - Kx_2 &= M_1 g \\
-B\dot{x}_1 - Kx_1 + M_2\ddot{x}_2 + B\dot{x}_2 + 3Kx_2 - Kx_3 &= M_2 g \\
-Kx_2 + M_3\ddot{x}_3 + Kx_3 &= M_3 g - f_a(t)
\end{aligned}
$$

(b) Letting $f_a(t) = 0$, replacing x_1, x_2, and x_3 by the constant displacements x_{1_0}, x_{2_0}, and x_{3_0}, and noting that all the derivatives of these constant displacements are zero, we have the three algebraic equations

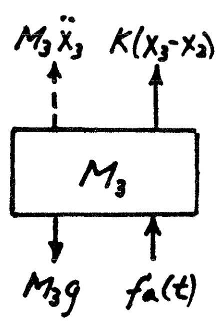


$$
2x_{1_0} - x_{2_0} = \frac{M_1 g}{K}, \quad -x_{1_0} + 3x_{2_0} - x_{3_0} = \frac{M_2 g}{K},
$$

$$
\text{and} \quad -x_{2_0} + x_{3_0} = \frac{M_3 g}{K}
$$

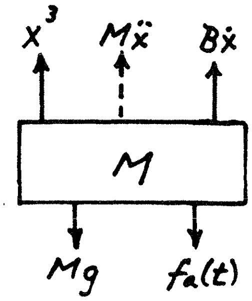
from which the four spring elongations are found to be

$$
\begin{aligned}
x_{1_0} &= (2M_1 + M_2 + M_3)g/3K \\
x_{2_0} &= (M_1 + 2M_2 + 2M_3)g/3K \\
x_{2_0} - x_{1_0} &= (-M_1 + M_2 + M_3)g/3K \\
x_{3_0} - x_{2_0} &= M_3 g/K
\end{aligned}
$$



2.19

(a) From the free-body diagram for the mass



$$M\ddot{x} + B\dot{x} + x^3 = f_a(t) + Mg$$

(b) With $f_a(t) = 0$ and with x replaced by the constant displacement x_0, the equation gives $x_0^3 = Mg$. Substituting $x = x_0 + z$, $\dot{x} = \dot{z}$, and $\ddot{x} = \ddot{z}$ into the differential equation, we have

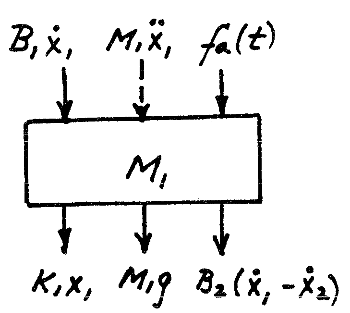
$$M\ddot{z} + B\dot{z} + (x_0^3 + 3x_0^2 z + 3x_0 z^2 + z^3) = f_a(t) + Mg$$

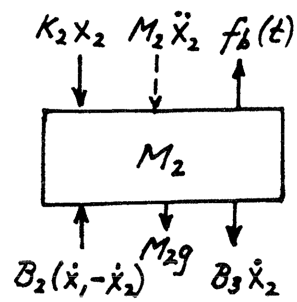
Canceling the terms x_0^3 and Mg yields

$$M\ddot{z} + B\dot{z} + 3x_0^2 z + 3x_0 z^2 + z^3 = f_a(t)$$

(c) Because of the nonlinear spring, the differential equation in z is not the same as the one in x with the Mg term deleted.

2.20

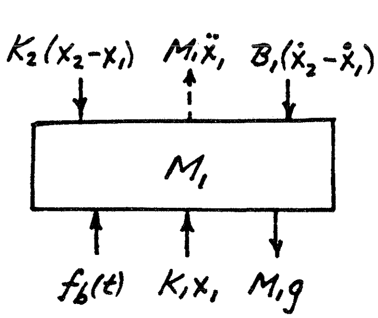


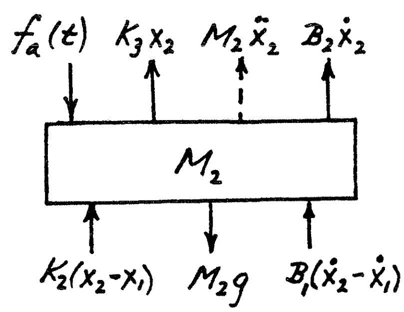


Summing the forces shown on each of the free-body diagrams and collecting terms, we get

$$\begin{aligned} M_1\ddot{x}_1 + (B_1 + B_2)\dot{x}_1 + K_1x_1 - B_2\dot{x}_2 &= -f_a(t) - M_1g \\ -B_2\dot{x}_1 + M_2\ddot{x}_2 + (B_2 + B_3)\dot{x}_2 + K_2x_2 &= f_b(t) - M_2g \end{aligned}$$

2.21



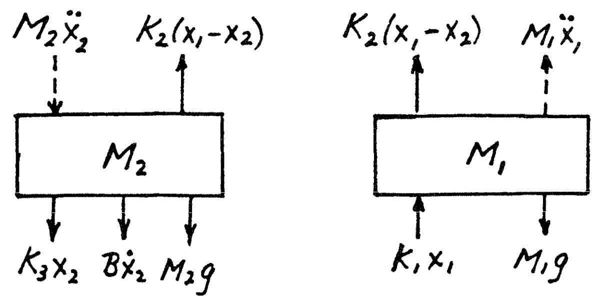


Summing the forces shown on each of the free-body diagrams and collecting terms, we get

$$\begin{aligned} M_1\ddot{x}_1 + B_1\dot{x}_1 + (K_1 + K_2)x_1 - B_1\dot{x}_2 - K_2x_2 &= -f_b(t) + M_1g \\ -B_1\dot{x}_1 - K_2x_1 + M_2\ddot{x}_2 + (B_1 + B_2)\dot{x}_2 + (K_2 + K_3)x_2 &= f_a(t) + M_2g \end{aligned}$$

2.22

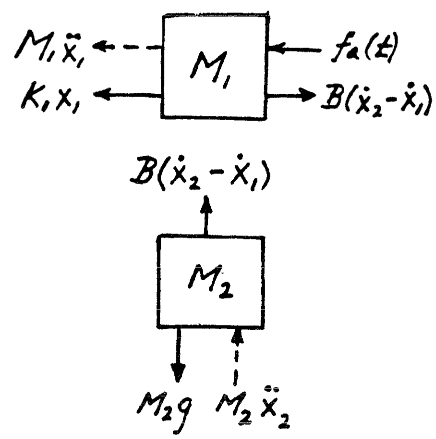
When drawing the free-body diagrams, note that the upward force of the cable on M_2 is the same as the upward force of the cable on M_1 because of the pulley. Summing the forces shown on each of the diagrams and collecting terms, we get



$$
\begin{aligned}
M_1\ddot{x}_1 + (K_1+K_2)x_1 - K_2x_2 &= M_1g \\
-K_2x_1 + M_2\ddot{x}_2 + B\dot{x}_2 + (K_2+K_3)x_2 &= -M_2g
\end{aligned}
$$

2.23

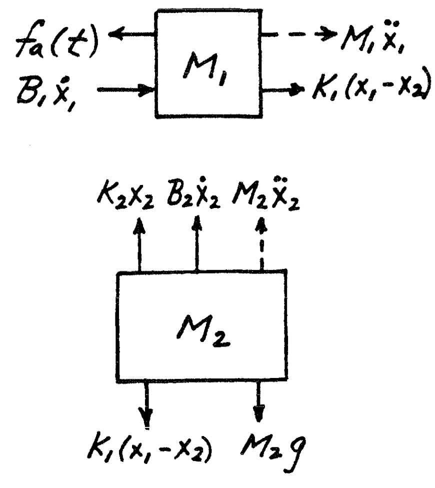
When drawing the free-body diagrams, note that the force of the cable to the right on M_1 is the same as the upward force of the cable on M_2 because of the pulley. Summing the forces shown on each of the diagrams and collecting terms, we get



$$
\begin{aligned}
M_1\ddot{x}_1 + B\dot{x}_1 + Kx_1 - B\dot{x}_2 &= -f_a(t) \\
-B\dot{x}_1 + M_2\ddot{x}_2 + B\dot{x}_2 &= M_2g
\end{aligned}
$$

2.24

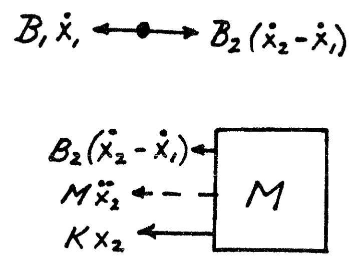
When drawing the free-body diagrams, note that the downward force of the cable on M_2 is the same as the force of the cable to the right on M_1 because of the pulley. Summing the forces shown on each of the diagrams and collecting terms, we get



$$
\begin{aligned}
M_1\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - K_1x_2 &= f_a(t) \\
-K_1x_1 + M_2\ddot{x}_2 + B_2\dot{x}_2 + (K_1+K_2)x_2 &= M_2g
\end{aligned}
$$

2.25

The free-body diagrams for the junction of the dashpots and for the mass are at the right. By D'Alembert's law,



$$\begin{aligned} B_1\dot{x}_1 - B_2(\dot{x}_2 - \dot{x}_1) &= 0 \\ B_2(\dot{x}_2 - \dot{x}_1) + Kx_2 + M\ddot{x}_2 &= 0 \end{aligned}$$

Rearranging these two equations, we have

$$\dot{x}_1 = \left(\frac{B_2}{B_1 + B_2}\right)\dot{x}_2 \qquad \text{(A)}$$

$$-B_2\dot{x}_1 + M\ddot{x}_2 + B_2\dot{x}_2 + Kx_2 = 0 \qquad \text{(B)}$$

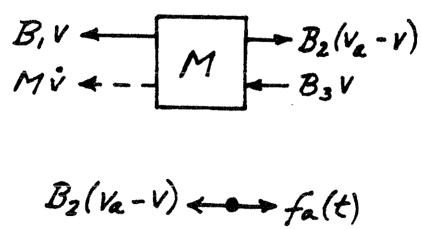
From (A), we see that $\dot{x}_1$ is proportional to $\dot{x}_2$. Substituting (A) into (B) and collecting terms, we get

$$M\ddot{x}_2 + \left(\frac{B_1B_2}{B_1 + B_2}\right)\dot{x}_2 + Kx_2 = 0$$

From this result, we see that $B_{\text{eq}} = B_1B_2/(B_1 + B_2)$ for the series combination of B_1 and B_2.

2.26

Let v_a denote the velocity, with the positive sense to the right, of the right end of the dashpot labeled B_2. From the two free-body diagrams,



$$M\dot{v} + (B_1 + B_3)v - B_2(v_a - v) = 0 \qquad \text{and} \qquad B_2(v_a - v) = f_a(t)$$

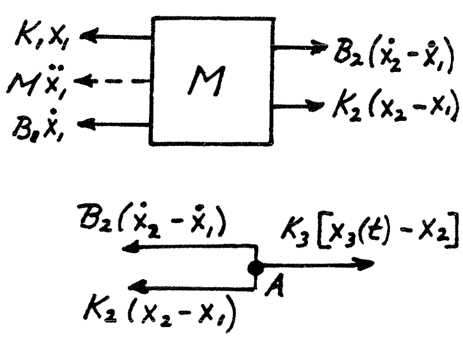
By combining the two equations, we have

$$M\dot{v} + (B_1 + B_3)v = f_a(t)$$

which indicates that the force $f_a(t)$ is transmitted through the dashpot B_2 and exerted on the mass M. The friction elements B_1 and B_3 are in parallel and can be replaced by $B_{\text{eq}} = B_1 + B_3$.

2.27

Because the motion of point B is known, we need free-body diagrams only for the mass M and point A. Summing the forces shown on each of these diagrams and collecting terms, we get

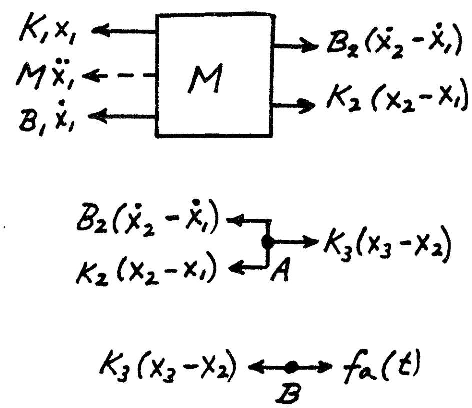


$$\begin{aligned} M\ddot{x}_1 + (B_1 + B_2)\dot{x}_1 + (K_1 + K_2)x_1 - B_2\dot{x}_2 - K_2x_2 &= 0 \\ -B_2\dot{x}_1 - K_2x_1 + B_2\dot{x}_2 + (K_2 + K_3)x_2 &= K_3x_3(t) \end{aligned}$$

2.28

Summing the forces shown on each of the free-body diagrams and collecting terms, we have

$$\begin{aligned} M\ddot{x}_1 + (B_1 + B_2)\dot{x}_1 + (K_1 + K_2)x_1 - B_2\dot{x}_2 - K_2x_2 &= 0 \\ -B_2\dot{x}_1 - K_2x_1 + B_2\dot{x}_2 + (K_2 + K_3)x_2 - K_3x_3 &= 0 \\ K_3(x_3 - x_2) &= f_a(t) \end{aligned}$$



The last two equations may be combined as

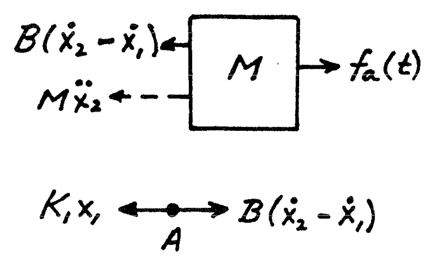
$$-B_2\dot{x}_1 - K_2x_1 + B_2\dot{x}_2 + K_2x_2 = f_a(t)$$

showing that $f_a(t)$ is transmitted through the spring K_2 to point A. This equation can be obtained directly from the free-body diagram for point A if we replace the force exerted by spring K_3 by $f_a(t)$. The differential equations governing the motion of the mass M and the point A are the first and fourth of this set.

2.29

Summing the forces shown on each of the free-body diagrams and collecting terms, we get

$$\begin{aligned} B\dot{x}_1 + Kx_1 - B\dot{x}_2 &= 0 \\ -B\dot{x}_1 + M\ddot{x}_2 + B\dot{x}_2 &= f_a(t) \end{aligned}$$



2.30

From the discussion associated with Figures 2.5 and 2.8, we note that the tensile forces on the ends of each element are $f_a(t)$. Thus the order of the elements does not affect the relationship between $f_a(t)$ and $x_5 - x_1$. From (37) and (38),

$$K_{\text{eq}} = \frac{K_1K_2}{K_1 + K_2} \qquad \text{and} \qquad B_{\text{eq}} = \frac{B_1B_2}{B_1 + B_2}$$

Alternatively, we may write

$$\begin{aligned} x_5 - x_1 &= (x_5 - x_4) + (x_4 - x_3) + (x_3 - x_2) + (x_2 - x_1) \\ \dot{x}_5 - \dot{x}_1 &= \frac{1}{B_2}f_a(t) + \frac{1}{K_2}\dot{f}_a + \frac{1}{B_1}f_a(t) + \frac{1}{K_1}\dot{f}_a = \frac{1}{K_{\text{eq}}}\dot{f}_a + \frac{1}{B_{\text{eq}}}f_a(t) \end{aligned}$$

where K_{eq} and B_{eq} are defined as before.

Chapter 3
Standard Forms for System Models

Topic	Section	Examples	Problems[1]
State-variable equations	1	1, 2, 3, 4, 5 6, 7, 8, 9	1, 2, 3*, 4, 5*, 6 7, 8, 9*, 10, 11*, 12 13*, 14, 15*, 16, 17 18*, 19, 20, 21*, 22
Input-output equations	2	10, 11, 12	23, 24*, 25, 26*, 27
Matrix formulation of state-variable models	3	13, 14, 15	28, 29*, 30, 31, 32

[1] The answers to problems with an asterisk are given in Appendix G.

3.1

(a) Replacing $\dot{x}$ by v in Equation (2.21), we have

$$M\dot{v} + Bv + Kx = f_a(t) + Mg$$

The state-variable and output equations are

$$\begin{aligned}
\dot{x} &= v \\
\dot{v} &= \frac{1}{M}[-Kx - Bv + f_a(t) + Mg] \\
w_K &= \frac{1}{2}Kx^2
\end{aligned}$$

(b) Replacing $\dot{z}$ by ω in Equation (2.24), we have

$$M\dot{w} + Bw + Kz = f_a(t)$$

The state-variable equation are

$$\begin{aligned}
\dot{z} &= w \\
\dot{w} &= \frac{1}{M}[-Kz - Bw + f_a(t)]
\end{aligned}$$

Using Equations (2.22) and (2.23), we write the output equation as

$$w_K = \frac{1}{2}K(x_0 + z)^2 = \frac{1}{2}K\left(\frac{Mg}{K} + z\right)^2$$

3.2

In the given equations, we replace $\dot{x}, \ddot{x}$, and $\dddot{x}$ by v, a, and $\dot{a}$, respectively. Then the state-variable and output equations are

$$\begin{aligned}
\dot{x} &= v \\
\dot{v} &= a \\
\dot{a} &= -\gamma x - \beta v - \alpha a + f_a(t) \\
y &= v + a
\end{aligned}$$

3.3

Defining $v = \dot{y}$, we rewrite the given equations as

$$\begin{aligned} \dot{v} + 4v + 2y &= x \\ \dot{x} + x + y &= f_a(t) \end{aligned}$$

Choosing y, v, and x as state variables, we have

$$\begin{aligned} \dot{y} &= v \\ \dot{v} &= -2y - 4v + x \\ \dot{x} &= -y - x + f_a(t) \end{aligned}$$

3.4

In the given equations, we replace $\dot{x}_1$ by v_1, $\ddot{x}_1$ by $\dot{v}_1$, $\dot{x}_2$ by v_2, and $\ddot{x}_2$ by $\dot{v}_2$. We then solve for $\dot{v}_1$ and $\dot{v}_2$ and write

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{v}_1 &= \frac{1}{M_1}[-K_1x_1 - Bv_1 + Bv_2] \\ \dot{x}_2 &= v_2 \\ \dot{v}_2 &= \frac{1}{M_2}[Bv_1 - K_2x_2 - Bv_2 + f_a(t)] \\ y &= v_1 - v_2 \end{aligned}$$

3.5

We rewrite (2.20) as

$$\begin{aligned} M\dot{v}_1 + (B_1 + B_2 + B_3)v_1 + K_1x_1 - B_2v_2 &= 0 \\ -B_2v_1 + M_2\dot{v}_2 + B_2v_2 + K_2x_2 &= f_a(t) \end{aligned}$$

Then the state-variable and output equations are

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{v}_1 &= \frac{1}{M_1}[-K_1x_1 + (B_1 + B_2 + B_3)v_1 + B_2v_2] \\ \dot{x}_2 &= v_2 \\ \dot{v}_2 &= \frac{1}{M_2}[B_2v_1 - K_2x_2 - Bv_2 + f_a(t)] \\ y_1 &= B_2(v_2 - v_1) \\ y_2 &= K_2x_2 \end{aligned}$$

3.6
In (2.19) we replace $\dot{x}$ by v_x, $\ddot{x}$ by $\dot{v}_x$, $\dot{z}$ by v_z, and $\ddot{z}$ by $\dot{v}_z$. We then solve (2.19) for $\dot{v}_x$ and $\dot{v}_z$ and write

$$\begin{aligned} \dot{x} &= v_x & \text{(A)} \\ \dot{v}_x &= \frac{1}{M_1}[-K_1 x - (B_1 + B_3)v_x + B_2 v_z] & \text{(B)} \\ \dot{z} &= v_z & \text{(C)} \\ \dot{v}_z &= \frac{1}{M_2}[-K_2 x - M_2\dot{v}_x - K_2 z - B_2 v_z + f_a(t)] & \text{(D)} \end{aligned}$$

Because (D) contains a derivative on the right side, we substitute (B) into (D) to obtain

$$\dot{v}_z = \frac{1}{M_1 M_2}[(M_2K_1 - M_1K_2)x + M_2(B_1 + B_3)v_x - M_1K_2 z - B_2(M_1 + M_2)v_z + M_1 f_a(t)] \quad \text{(E)}$$

The state-variable equations are (A), (B), (C), and (E). The output equations are $y_1 = B_2 v_z$ and $y_2 = K_2(x + z)$.

3.7 With $\dot{x}_1 = v_1$ and $\dot{x}_2 = v_2$, we rewrite (2.25) as

$$\begin{aligned} M_1\dot{v}_1 + K_1 x_1 - Bv_2 - (K_2 + K_3)x_2 &= M_1 g + f_a(t) \\ M_2\dot{v}_1 + M_2\dot{v}_2 + Bv_2 + (K_2 + K_3)x_2 &= M_2 g \end{aligned}$$

Solving these two equations for $\dot{v}_1$ and $\dot{v}_2$, respectively, we have

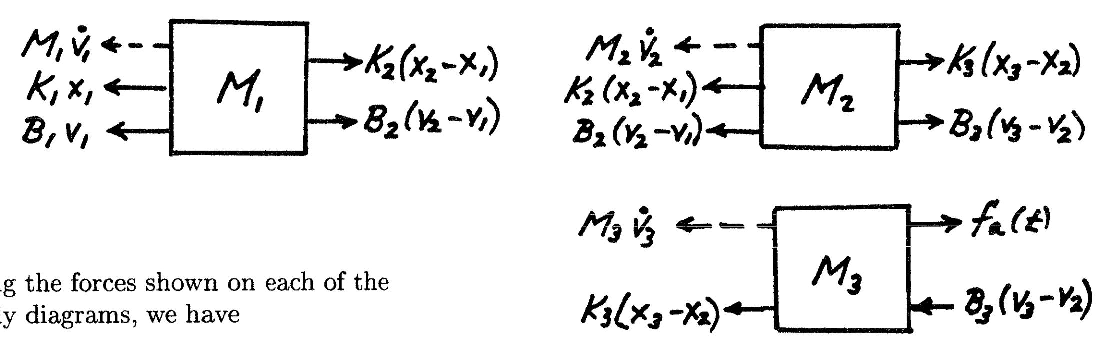
$$\begin{aligned} \dot{v}_1 &= \frac{1}{M_1}[-K_1 x_1 + (K_2 + K_3)x_2 + Bv_2 + M_1 g + f_a(t)] & \text{(A)} \\ \dot{v}_2 &= \frac{1}{M_2}[-M_2\dot{v}_1 - (K_2 + K_3)x_2 - Bv_2 + M_2 g] & \text{(B)} \end{aligned}$$

Substituting (A) into (B) and simplifying the result, we obtain

$$\dot{v}_2 = \frac{1}{M_1 M_2}[K_1 M_2 x_1 - (K_2 + K_3)(M_1 + M_2)x_2 - B(M_1 + M_2)v_2 - M_2 f_a(t)] \quad \text{(C)}$$

The state-variable equation are $\dot{x}_1 = v_1$ and $\dot{x}_2 = v_2$, plus (A) and (C). The output equations are $f_{K_1} = K_1 x_1$, $f_{K_2} = K_2 x_2$, and $f_{K_3} = -K_3 x_2$.

3.8



Summing the forces shown on each of the free-body diagrams, we have

$$\begin{aligned} M_1\dot{v}_1 + (B_1 + B_2)v_1 + (K_1 + K_2)x_1 - B_2 v_2 - K_2 x_2 &= 0 \\ -B_2 v_1 - K_2 x_1 + M_2\dot{v}_2 + (B_2 + B_3)v_2 + (K_2 + K_3)x_2 - B_3 v_3 - K_3 x_3 &= 0 \\ -B_3 v_2 - K_3 x_2 + M_3\dot{v}_3 + B_3 v_3 + K_3 x_3 &= f_a(t) \end{aligned}$$

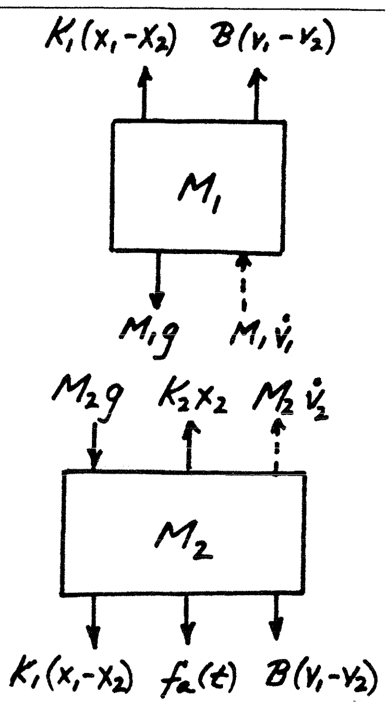
The state-variable equations are

$$
\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{v}_1 &= \frac{1}{M_1}[-(K_1 + K_2)x_1 - (B_1 + B_2)v_1 + K_2x_2 + B_2v_2] \\
\dot{x}_2 &= v_2 \\
\dot{v}_2 &= \frac{1}{M_2}[K_2x_1 + B_2v_1 - (K_2 + K_3)x_2 - (B_2 + B_3)v_2 + K_3x_3 + B_3v_3] \\
\dot{x}_3 &= v_3 \\
\dot{v}_3 &= \frac{1}{M_3}[K_3x_2 + B_3v_2 - K_3x_3 - B_3v_3 + f_a(t)]
\end{aligned}
$$

The output equation is $m_T = M_1v_1 + M_2v_2 + M_3v_3$.

3.9

We sum the forces shown on each of the free-body diagrams, where v_1 and v_2 denote the downward velocities of M_1 and M_2, respectively.



$$
\begin{aligned}
M_1\dot{v}_1 + Bv_1 + K_1x_1 - Bv_2 - K_1x_2 &= M_1g \\
-Bv_1 - K_1x_1 + M_2\dot{v}_2 + Bv_2 + (K_1 + K_2)x_2 &= M_2g + f_a(t)
\end{aligned}
$$

The state-variable equations are

$$
\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{v}_1 &= \frac{1}{M_1}[-K_1x_1 - Bv_1 + K_2x_2 + Bv_2 + M_1g] \\
\dot{x}_2 &= v_2 \\
\dot{v}_2 &= \frac{1}{M_2}[K_1x_1 + Bv_1 - (K_1 + K_2)x_2 - Bv_2 + M_2g + f_a(t)]
\end{aligned}
$$

The output equations are

$$
\begin{aligned}
y_1 &= x_1 - x_2 \\
a_1 &= \frac{1}{M_1}[-K_1x_1 - Bv_1 + K_2x_2 + Bv_2 + M_1g]
\end{aligned}
$$

3.10

Rewriting (2.20) with $\dot{x}_1 = v_1$, $\dot{x}_2 = v_2$, and $K_1 = 0$, we have

$$
\begin{aligned}
M_1\dot{v}_1 + (B_1 + B_2 + B_3)v_1 - B_2v_2 &= 0 \\
-B_2v_1 + M_2\dot{v}_2 + B_2v_2 + K_2x_2 &= f_a(t)
\end{aligned}
$$

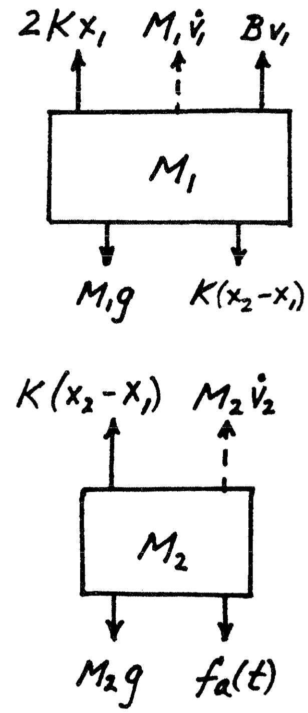
Solving these equations for $\dot{v}_1$ and $\dot{v}_2$, we obtain the following state-variable equations:

$$
\begin{aligned}
\dot{v}_1 &= \frac{1}{M_1}[-(B_1 + B_2 + B_3)v_1 + B_2v_2] \\
\dot{x}_2 &= v_2 \\
\dot{v}_2 &= \frac{1}{M_2}[B_2v_1 - K_2x_2 - B_2v_2 + f_a(t)]
\end{aligned}
$$

The output equations are $y_1 = B(v_1 - v_2)$ and $y_2 = K_2x_2$. The energy stored in M_1, M_2, and K_2 does not depend on the displacement x_1. Also, x_1 does not appear in either output equation, so it is not needed as a state variable.

3.11

We sum the forces shown on each of the free-body diagrams, where v_1 and v_2 denote the downward velocities of M_1 and M_2, respectively.



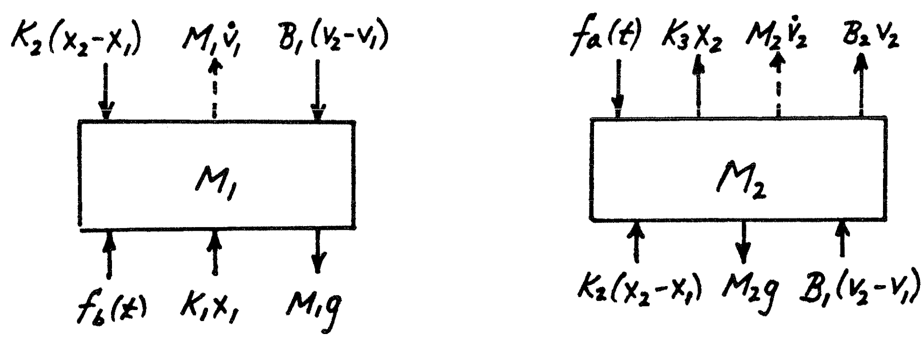
$$
\begin{aligned}
M_1\dot{v}_1 + Bv_1 + 3Kx_1 - Kx_2 &= M_1g \\
-Kx_1 + M_2\dot{v}_2 + Kx_2 &= M_2g + f_a(t)
\end{aligned}
$$

Solving these equations for $\dot{v}_1$ and $\dot{v}_2$, we have the following state-variable equations:

$$
\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{v}_1 &= \frac{1}{M_1}[-3Kx_1 - Bv_1 + Kx_2 + M_1g] \\
\dot{x}_2 &= v_2 \\
\dot{v}_2 &= \frac{1}{M_2}[Kx_1 - Kx_2 + M_2g + f_a(t)]
\end{aligned}
$$

The output equation is $y = x_2 - x_1$.

3.12



We sum the forces shown on each of the free-body diagrams, where v_1 and v_2 denote the downward velocities of M_1 and M_2, respectively.

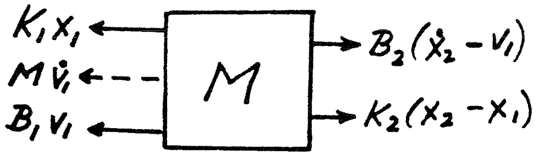
$$
\begin{aligned}
M_1\dot{v}_1 + B_1v_1 + (K_1 + K_2)x_1 - B_1v_2 - K_2x_2 &= -f_b(t) + M_1g \\
-B_1v_1 - K_2x_1 + M_2\dot{v}_2 + (B_1 + B_2)v_2 + (K_2 + K_3)x_2 &= f_a(t) + M_2g
\end{aligned}
$$

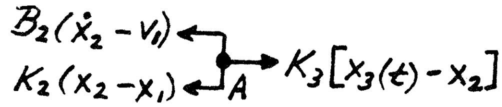
Solving these equations for $\dot{v}_1$ and $\dot{v}_2$, we have the following state-variable equations:

$$
\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{v}_1 &= \frac{1}{M_1}[-(K_1 + K_2)x_1 - B_1v_1 + K_2x_2 + B_1v_2 - f_b(t) + M_1g] \\
\dot{x}_2 &= v_2 \\
\dot{v}_2 &= \frac{1}{M_2}[K_2x_1 + B_1v_1 - (K_2 + K_3)x_2 - (B_1 + B_2)v_2 + f_a(t) + M_2g]
\end{aligned}
$$

The output equation is $y = x_1 - x_2$. Because the elongations of the springs are not all independent, there are only four state variables even though there are five energy-storing elements.

3.13





We sum the forces shown on each of the free-body diagrams, where $v_1 = \dot{x}_1$.

$$\begin{aligned} M\dot{v}_1 + (B_1 + B_2)v_1 + (K_1 + K_2)x_1 - B_2\dot{x}_2 - K_2x_2 &= 0 \\ -B_2v_1 - K_2x_1 + B_2\dot{x}_2 + (K_2 + K_3)x_2 &= K_3x_3(t) \end{aligned}$$

Solving these equations for $\dot{v}_1$ and $\dot{x}_2$, we have

$$\begin{aligned} \dot{x}_1 &= v_1 && \text{(A)} \\ \dot{v}_1 &= \frac{1}{M}[-(K_1 + K_2)x_1 - (B_1 + B_2)v_1 + K_2x_2 + B_2\dot{x}_2] && \text{(B)} \\ \dot{x}_2 &= \frac{1}{B_2}[K_2x_1 + B_2v_1 - (K_2 + K_3)x_2 + K_3x_3(t)] && \text{(C)} \end{aligned}$$

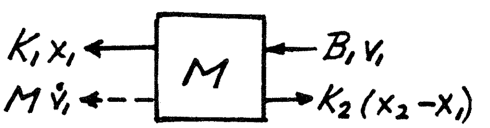
Substituting (C) into (B), in order to eliminate the derivative on the right side, and simplifying the result, we obtain

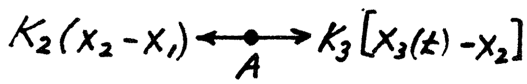
$$\dot{v}_1 = \frac{1}{M}[-K_1x_1 - B_1v_1 - K_3x_2 + K_3x_3(t)] \qquad \text{(D)}$$

The state-variable equations are (A), (C), and (D). The output is given by $y = B_2(\dot{x}_2 - v_1)$. Inserting (C) into this expression and simplifying the result give the output equation

$$y = K_2x_1 - (K_2 + K_3)x_2 + K_3x_3(t)$$

3.14





We sum the forces shown on each of the free-body diagrams, where $v_1 = \dot{x}_1$.

$$\begin{aligned} M\dot{v}_1 + B_1v_1 + (K_1 + K_2)x_1 - K_2x_2 &= 0 && \text{(A)} \\ -K_2x_1 + (K_2 + K_3)x_2 &= K_3x_3(t) && \text{(B)} \end{aligned}$$

From (B), we have the output equation

$$x_2 = \frac{K_2x_1 + K_3x_3(t)}{K_2 + K_3} \qquad \text{(C)}$$

Substituting (C) into (A), solving for $\dot{v}_1$, and noting that $\dot{x}_1 = v_1$, we obtain the following state-variable equations:

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{v}_1 &= \frac{1}{M}\left[-\left(K_1 + \frac{K_2K_3}{K_2 + K_3}\right)x_1 - B_1v_1 + \left(\frac{K_2}{K_2 + K_3}\right)x_3(t)\right] \end{aligned}$$

Because K_2 and K_3 are in series, there are only two state variables even though there are three energy-storing elements.

3.15

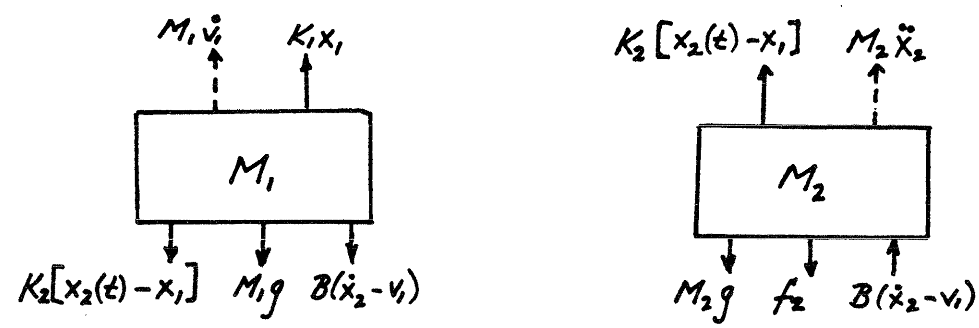
The first two equations can be rewritten as

$$\begin{aligned}
\frac{d}{dt}[x_1 - 2u_2(t)] &= -3x_1 + 2x_2 + u_1(t) \\
\frac{d}{dt}[x_2 - u_1(t)] &= 2x_1 + x_2
\end{aligned}$$

We define two new state variables as $q_1 = x_1 - 2u_2(t)$ and $q_2 = x_2 - u_1(t)$. Note that $x_1 = q_1 + 2u_2(t)$ and $x_2 = q_2 + u_1(t)$. Substituting these expressions into (A) and into the given output equation, we obtain

$$\begin{aligned}
\dot{q}_1 &= -3q_1 + 2q_2 + 3u_1(t) - 6u_2(t) \\
\dot{q}_2 &= 2q_1 + q_2 + u_1(t) + 4u_2(t) \\
y &= q_1 - q_2 - u_1(t) + 3u_2(t)
\end{aligned}$$

3.16



Because the motion of M_2 is known, we need only the free-body diagram for M_1 in order to find the state-variable equations. Summing the forces shown on that diagram, where $\dot{x}_1 = v_1$, we have

$$M_1\dot{v}_1 + Bv_1 + (K_1 + K_2)x_1 = B\dot{x}_2 + K_2x_2(t) + M_1g \qquad \text{(A)}$$

Because of the derivative of the input on the right side, solving for $\dot{v}_1$ will not yield an appropriate state-variable equation. Hence we rewrite (A) as

$$M_1\frac{d}{dt}\left[v_1 - \frac{B}{M_1}x_2(t)\right] = -(K_1 + K_2)x_1 - Bv_1 + K_2x_2(t) + M_1g \qquad \text{(B)}$$

We now define the new state variable $q = v_1 - (B/M_1)x_2(t)$. Then $v_1 = q + (B/M_1)x_2(t)$. Substituting these expressions into (B), we obtain the following state-variable equations:

$$\begin{aligned}
\dot{x}_1 &= q + \frac{B}{M_1}x_2(t) && \text{(C)} \\
\dot{q} &= \frac{1}{M_1}\left[-(K_1 + K_2)x_1 - Bq + \left(K_2 - \frac{B^2}{M_1}\right)x_2(t) + M_1g\right] && \text{(D)}
\end{aligned}$$

In order to obtain an output equation for f_2, we sum the forces shown on the free-body diagram for M_2.

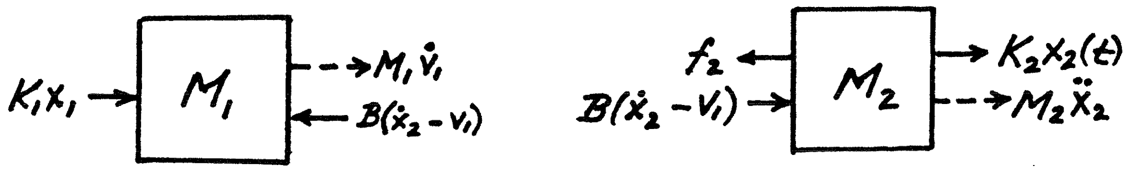
$$f_2 = -Bv_1 - K_2x_1 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2(t) - M_2g$$

Replacing v_1 by the expression in (C), we get

$$f_2 = -K_2 x_1 - Bq + M_2\ddot{x}_2 + B\dot{x}_2 + \left(K_2 - \frac{B^2}{M_1}\right) x_2(t) - M_2 g$$

In this situation, it is not possible to obtain an output equation that does not contain derivatives of the input.

3.17



Because the motion of M_2 is known, we need only the free-body diagram for M_1 in order to find the state-variable equations. Summing the forces shown on that diagram, where $\dot{x}_1 = v_1$, we have

$$M_1\dot{v}_1 + Bv_1 + K_1 x_1 = B\dot{x}_2 \qquad \text{(A)}$$

Because of the derivative of the input on the right side, solving for $\dot{v}_1$ will not yield an appropriate state-variable equation. Hence we rewrite (A) as

$$M_1 \frac{d}{dt}\left[v_1 - \frac{B}{M_1}x_2(t)\right] = -K_1 x_1 - Bv_1 \qquad \text{(B)}$$

We now define the new state variable $q = v_1 - (B/M_1)x_2(t)$ and note that $v_1 = q + (B/M_1)x_2(t)$. Substituting these expressions into (B), we obtain the following state-variable equations:

$$\begin{aligned} \dot{x}_1 &= q + \frac{B}{M_1}x_2(t) && \text{(C)} \\ \dot{q} &= \frac{1}{M_1}\left[-K_1 x_1 - Bq - \frac{B^2}{M_1}x_2(t)\right] && \text{(D)} \end{aligned}$$

In order to obtain an output equation for f_2, we sum the forces shown on the free-body diagram for M_2.

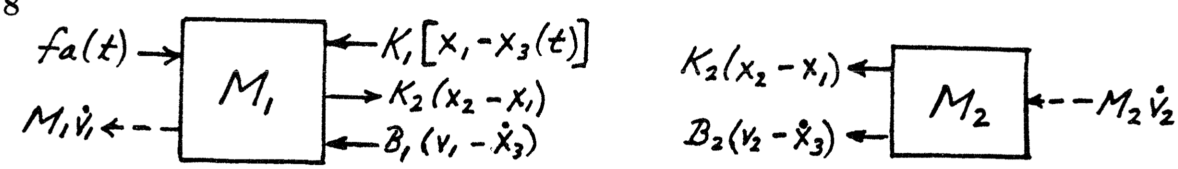
$$f_2 = -Bv_1 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2 x_2(t)$$

Replacing v_1 by the expression in (C), we get

$$f_2 = -Bq + M_2\ddot{x}_2 + B\dot{x}_2 + \left(K_2 - \frac{B^2}{M_1}\right) x_2(t)$$

In this situation, it is not possible to obtain an output equation that does not contain derivatives of the input.

3.18



Because the motion of M_3 is known, we need only the free-body diagram for M_1 and M_2 in order to find the state-variable equations. Summing the forces shown on these diagrams, where $v_1 = \dot{x}_1$, and $v_2 = \dot{x}_2$, we have

$$\begin{aligned} M_1\dot{v}_1 + B_1v_1 + (K_1 + K_2)x_1 - K_2x_2 &= B_1\dot{x}_3 + K_1x_3(t) + f_a(t) \\ -K_2x_1 + M_2\dot{v}_2 + B_2v_2 + K_2x_2 &= B_2\dot{x}_3 \end{aligned}$$

Because of the derivative of the input on the right sides of these equations, solving for $\dot{v}_1$ and $\dot{v}_2$ will not yield appropriate state-variable equations. Hence we rewrite (A) as

$$\begin{aligned} M_1\frac{d}{dt}\left[v_1 - \frac{B_1}{M_1}x_3(t)\right] &= -(K_1 + K_2)x_1 - B_1v_1 + K_2x_2 + K_1x_3(t) + f_a(t) \\ M_2\frac{d}{dt}\left[v_2 - \frac{B_2}{M_2}x_3(t)\right] &= K_2x_1 - K_2x_2 - B_2v_2 \end{aligned}$$

We now define the new state variables $q_1 = v_1 - (B_1/M_1)x_3(t)$ and $q_2 = v_2 - (B_2/M_2)x_3(t)$. Then

$$v_1 = q_1 + \frac{B_1}{M_1}x_3(t) \qquad \text{and} \qquad v_2 = q_2 + \frac{B_2}{M_2}x_3(t) \tag{C}$$

Substituting these expressions into (B), we obtain the following state-variable equations:

$$\begin{aligned} \dot{x}_1 &= q + \frac{B_1}{M_1}x_3(t) \\ \dot{q}_1 &= \frac{1}{M_1}\left[-(K_1 + K_2)x_1 - B_1q_1 + K_2x_2 + \left(K_1 - \frac{B_1^2}{M_1}\right)x_3(t) + f_a(t)\right] \\ \dot{x}_2 &= q_2 + \frac{B_2}{M_2}x_3(t) \\ \dot{q}_2 &= \frac{1}{M_2}[K_2x_1 - K_2x_2 - B_2q_2 - \frac{B_2^2}{M_2}x_3(t)] \end{aligned}$$

The output equations are given in (C).

3.19

In order to retain x_1 and x_2 as state variables, we can solve simultaneously the first two equations in the problem statement for $\dot{x}_1$ and $\dot{x}_2$. Adding twice the second equation to the first and dividing the result by 3, we get

$$\dot{x}_1 = \frac{7}{3}x_1 + 2x_2 - u(t)$$

Subtracting the second equation from the first and then dividing by 3 give

$$\dot{x}_2 = \frac{1}{3}x_1 + x_2 - 2u(t)$$