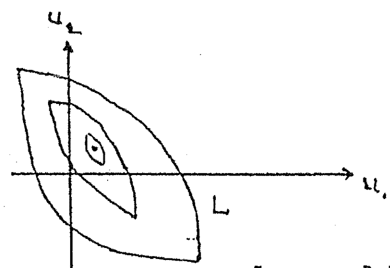
Chapter 1

1.1-1

a).



$$u^* = -Q^{-1}S = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$L_{uu} = Q = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}.$$

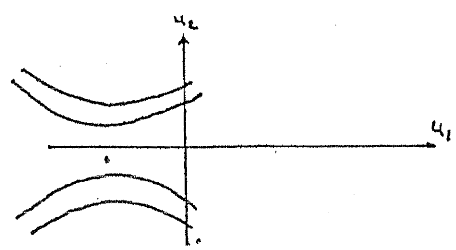
Since Q is negative definite, $L_{uu} < 0$. Therefore, u^* is local maximum.

$$L_u = \frac{1}{2}[u_1 \quad u_2]\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + [0 \quad 1]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\frac{1}{2}u_1^2 + u_1u_2 - u_2^2 + u_2$$

Optimum value of L, $L^* = -1/2$. The gradient

$$L_u = Qu + S = \begin{bmatrix} u_2 - u_1 \\ u_1 - 2u_2 + 1 \end{bmatrix}.$$

b).



$$u^* = -Q^{-1}S = -\frac{1}{3}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

$$L_{uu} = Q = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Since $|Q| \not< 0$, u^* is a saddle point.

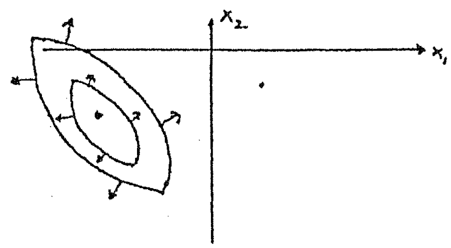
1.1-2

$$L(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2 + 3x_1$$

$$\left.\begin{array}{r}\frac{\partial L}{\partial x_1} = 2x_1 - x_2 + 3 = 0 \\ \frac{\partial L}{\partial x_2} = -x_1 + 2x_2 = 0\end{array}\right\} \Rightarrow x_1^* = -2, \quad x_2^* = -1.$$

$$L = -\frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$L_{uu} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} > 0.$$



1.1-3

a). $u = \begin{bmatrix} x \\ y \end{bmatrix}, \quad f_u = \begin{bmatrix} 2x \\ 4y^3 \end{bmatrix}, \quad f_u|_{u=0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

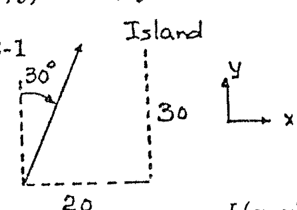
b).

$$f_{uu} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix}, \quad f_{uu}|_{u=0} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

$|f_{uu}| = 0 \Rightarrow$ origin is a singular point.

c). $f(x,y) = x^2 + y^4$. $f(0,0) = 0$, $x^2 > 0$ for all $x \neq 0$. $y^4 > 0$ for all $y \neq 0$. $f(x,y) = x^2 + y^4 > 0$ for all $x \neq 0$ and $y \neq 0 \Rightarrow (0,0)$ is a minimum point.

1.2-1



$$L(x,y) = \frac{1}{2}(x-20)^2 + \frac{1}{2}(y-30)^2$$

$$f(x,y) = \sqrt{3}x - y = 0,$$

$$H(x,y) = L + \lambda^T f = \frac{1}{2}(x-20)^2 + \frac{1}{2}(y-30)^2 + \lambda(\sqrt{3}x - y),$$

$$\left.\begin{array}{r} H_\lambda = \sqrt{3}x - y = 0 \\ H_x = x - 20 + \lambda\sqrt{3} = 0 \\ H_y = y - 30 - \lambda = 0 \end{array}\right\} \Rightarrow \begin{array}{r} \lambda = 1.16 \\ x = 17.99 \\ y = 31.16 \end{array}$$

Closest point: $(x,y) = (17.99, 31.16)$, $distance = \sqrt{2L^*} = 2.32$.

Time $= \frac{\sqrt{x^2+y^2}}{10mph} \sim 36$ hours.

1.2-2

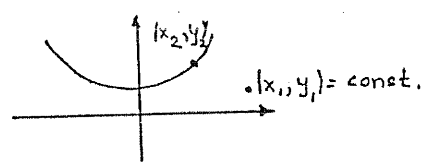
$$L = \min_{x,y}(d_1^2),$$
$$f = d_1^2 - d_2^2.$$

$$\begin{aligned} H &= H(x,y,\lambda) = (x-x_1)^2 + (y-y_1)^2 \\ &+\lambda\left[(x-x_1)^2 + (y-y_1)^2 - (x-x_2)^2 - (y-y_2)^2\right] \end{aligned}$$

$$\begin{aligned} \frac{\partial H}{\partial x} &= 2(x-x_1) + 2\lambda(x-x_1) - 2\lambda(x-x_2) = 0 \\ \frac{\partial H}{\partial y} &= 2(y-y_1) + 2\lambda(y-y_1) - 2\lambda(y-y_2) = 0 \\ \frac{\partial H}{\partial \lambda} &= (x-x_1)^2 + (y-y_1)^2 - (x-x_2)^2 - (y-y_2)^2 = 0 \end{aligned}$$

$$x = (1+\lambda)x_1 - \lambda x_2$$
$$y = (1+\lambda)y_1 - \lambda y_2$$
$$\lambda = -y_2, \quad x = \frac{x_1+x_2}{2}, \quad y = \frac{y_1+y_2}{2}$$

1.2-3



$$\begin{aligned} H &= \frac{1}{2}(x_1-x_2)^2 + \frac{1}{2}(y_1-y_2)^2 + \lambda\left(\frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} - 1\right) \\ H_{x_2} &= -(x_1-x_2) + \frac{2\lambda x_2}{a^2} = 0, \\ H_{y_2} &= -(y_1-y_2) + \frac{2\lambda y_2}{b^2} = 0, \\ H_\lambda &= \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} - 1 = 0. \end{aligned}$$

$$\lambda = \frac{a^2(x_1-x_2)}{2x_2} = -\frac{b^2(y_1-y_2)}{2y_2}.$$

From the above equation we obtain

$$\begin{aligned} \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} &= 1, \\ a^2x_1y_2 + b^2y_1x_2 &= (a^2+b^2)x_2y_2. \end{aligned}$$

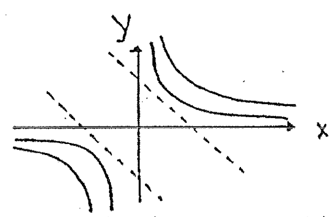
The above equations have two unknowns which can then be solved for.

1.2-4
a).

$$
\begin{aligned}
f(x,y) &= x^2+3x-y-6=0\\
L(x,y) &= \frac{1}{2}(x-2)^2+\frac{1}{2}(y-2)^2\\
H(x,y) &= L+\lambda^T f=\frac{1}{2}(x-2)^2+\frac{1}{2}(y-2)^2+\lambda(x^2+3x-y-6)\\
H_\lambda &= x^2+3x-y-6=0\\
H_x &= x-2+\lambda(2x+3)=0\\
H_y &= y-2-\lambda\\
y &= \frac{3x+8}{2x+3},\quad \lambda=\frac{2-x}{2x+3},\quad 2x^3+9x^2-6x-26=0,
\end{aligned}
$$

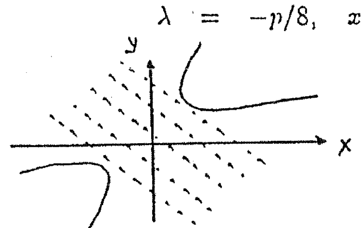
b). By Newton's method ($x_0=1.5$, four iterations). $x\sim 1.7086$, $y\sim 2.0454$, $\lambda\sim 0.0454$, distance=$\sqrt{2L^*}\sim 0.295$.

1.2-5
a).



$$
\begin{aligned}
H(x,y) &= xy+\lambda(2x+2y-p)\\
H_\lambda &= 2x+2y-p=0\\
H_x &= y+2\lambda=0\\
H_y &= x+2\lambda=0\\
\lambda &= -p/8,\quad x=y=p/4.
\end{aligned}
$$

b).



$$
\begin{aligned}
H(x,y) &= 2x+2y+\lambda(xy-a^2)\\
H_\lambda &= xy-a^2=0\\
H_x &= 2+\lambda y=0\\
H_y &= 2+\lambda x=0\\
\lambda &= \frac{\sqrt{2}}{a},\quad x=y=\sqrt{2}a.
\end{aligned}
$$

1.2-6

$$
\begin{aligned}
L &= \frac{1}{2}x^T\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}x+\frac{1}{2}u^T\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}u, \quad Q=\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}, \quad R=\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix},\\
x &= \begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}2 & 2\\1 & 0\end{bmatrix}u,\\
f &= x+\begin{bmatrix}-2 & -2\\-1 & 0\end{bmatrix}u+\begin{bmatrix}-1\\-3\end{bmatrix}=0, \quad B=\begin{bmatrix}-2 & -2\\-1 & 0\end{bmatrix}, \quad C=\begin{bmatrix}-1\\-3\end{bmatrix},\\
H &= \frac{1}{2}x^T\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}x+\frac{1}{2}u^T\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}u+\lambda^T\left\{x+\begin{bmatrix}-2 & -2\\-1 & 0\end{bmatrix}u+\begin{bmatrix}-1\\-3\end{bmatrix}\right\},\\
H_\lambda &= x+\begin{bmatrix}-2 & -2\\-1 & 0\end{bmatrix}u+\begin{bmatrix}-1\\-3\end{bmatrix}=0,\\
H_x &= \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}x+\lambda=0,\\
H_y &= \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}u+\begin{bmatrix}-2 & -2\\-1 & 0\end{bmatrix}\lambda=0,\\
u^* &= -(R+B^TQB)^{-1}B^TQC=\begin{bmatrix}-3\\13/5\end{bmatrix}, \quad K=\begin{bmatrix}0 & 1\\-0.4 & 1\end{bmatrix},\\
x^* &= -(I-BK)C=\begin{bmatrix}1.8\\6\end{bmatrix}, \quad \lambda^*=-Qx=\begin{bmatrix}-1.8\\-12\end{bmatrix},\\
L^* &= \frac{1}{2}C^T\lambda=\begin{bmatrix}0.9\\1.8\end{bmatrix}.
\end{aligned}
$$

1.2-7

a).

$$
\begin{aligned}
K &= \left(R+B^TQB\right)^{-1}B^TQ\\
U^* &= -KC\\
x^* &= -(I-BK)C\\
\lambda^* &= Q(I-BK)C\\
L^* &= \frac{1}{2}C^TQ[I-BK]C
\end{aligned}
$$

b).

$$
\begin{aligned}
S_0 &= Q-QB(B^TQB+R^{-1})B^TQ=Q(I-BK)\\
K &= \left(R+B^TQB\right)^{-1}B^TQ\\
0 &= B^TQ-\left(R+B^TQB\right)K
\end{aligned}
$$

Furthermore,

$$\begin{aligned} B^T Q(I-BK) - RK &= 0 \\ -K^T B^T Q(I-BK) + K^T RK &= 0 \\ Q(I-BK) - K^T B^T Q(I-BK) + K^T RK &= Q(I-BK) \end{aligned}$$

where $K^T B^T Q(I-BK) + K^T RK = 0$. So,

$$\begin{aligned} S_0 &= Q(I-BK) = (I-K^T B^T)Q(I-BK) + K^T RK \\ &= (I-BK)^T Q(I-BK) + K^T RK \\ L^* &= \frac{1}{2}C^T S_0 C = \frac{1}{2}C^T (I-BK)^T \sqrt{Q}^T \sqrt{(Q)}(I-BK)C \\ &\quad + \frac{1}{2}C^T K^T \sqrt{R}^T \sqrt{R}KC \\ &= \frac{1}{2}\left[\sqrt{Q}(I-BK)C\right]^T \left[\sqrt{Q}(I-BK)C\right] + \frac{1}{2}\left(\sqrt{R}KC\right)^T \sqrt{R}KC \end{aligned}$$

c).

$$\begin{aligned} S_0 &= Q - QB\left(R + B^T QB\right)^{-1} B^T Q \\ &= \left(Q^{-1} + BR^{-1}B^T\right)^{-1} \end{aligned}$$

1.2-8

a).

$$\begin{aligned} L &= x^2 y^2 z^2, \quad f = x^2 + y^2 + z^2 - r^2 = 0 \\ H &= L + \lambda^T f = x^2 y^2 z^2 + \lambda^T (x^2 + y^2 + z^2 - r^2) \\ H_\lambda &= x^2 + y^2 + z^2 - r^2 = 0 \\ H_x &= 2xy^2 z^2 + 2xy = 0 \qquad (1) \\ H_y &= 2x^2 y z^2 + 2\lambda y = 0 \qquad (2) \\ H_z &= 2x^2 y^2 z + 2z\lambda = 0 \qquad (3) \end{aligned}$$

From (1)-(3), we have $x^2 = y^2 = z^2$. Now,

$$\begin{aligned} H_\lambda &= 3x^2 - r^2 = 0 \Rightarrow x^2 = \frac{r^2}{3} \\ L^* &= \left(\frac{r^2}{3}\right)^3 \end{aligned}$$

b).

$$H = x^2 + y^2 + z^2 + \lambda\left(x^2 y^2 z^2 - \left(\frac{r^2}{3}\right)^3\right)$$

$$
\begin{aligned}
H_\lambda &= x^2y^2z^2 - \left(\frac{r^2}{3}\right)^3 = 0 \\
H_x &= 2x + 2x\lambda y^2z^2 = 0 \Rightarrow y^2z^2 = -1/\lambda \\
H_y &= 2y + 2y\lambda x^2z^2 = 0 \Rightarrow x^2z^2 = -1/\lambda \\
H_z &= 2z + 2z\lambda x^2y^2 = 0 \Rightarrow x^2y^2 = -1/\lambda.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
x^2 &= y^2 = z^2 \Rightarrow x^2 = r^2/3 \\
L^* &= x^2 + y^2 + z^2 = r^2.
\end{aligned}
$$

c).

$$
\begin{aligned}
L &= b_1^2b_2^2b_3^2\cdots b_n^2 \\
f &= b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2 - r^2 = 0 \\
H &= L + \lambda f \\
H_\lambda &= b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2 - r^2 = 0 \qquad (4) \\
H_{b_1} &= 2b_1b_2^2b_3^2\cdots b_n^2 + 2b_1\lambda = 0 \qquad (5) \\
&\vdots \\
H_{b_n} &= 2b_1^2b_2^2b_3^2\cdots b_n + 2b_n\lambda = 0 \qquad (6)
\end{aligned}
$$

From (4)-(6), we have

$$
\begin{aligned}
b_1^2 &= b_2^2 = b_3^2 = \cdots = b_n^2 \\
H_\lambda &= nb_1^2 - r^2 = 0 \Rightarrow b_1^2 = r^2/n \\
L^* &= \left(\frac{r^2}{n}\right)^n = \left(\frac{b_1^2 + b_2^2 + \cdots + b_n^2}{n}\right)^n \\
b_1^2b_2^2\cdots b_n^2 &\leq \left(\frac{b_1^2 + b_2^2 + \cdots + b_n^2}{n}\right)^n
\end{aligned}
$$

Let $a_i = b_i^2$. Hence, $a_i > 0$ and $(\frac{a_1+a_2+\cdots+a_n}{n})^n$ is the maximum value of $a_1a_2a_3\cdots a_n$ *i.e.*,

$$
\begin{aligned}
a_1a_2a_3\cdots a_n &\leq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n \\
(a_1a_2a_3\cdots a_n)^{\frac{1}{n}} &\leq \frac{(a_1 + a_2 + \cdots + a_n)}{n}
\end{aligned}
$$

1.2-9

$$\begin{aligned} f_1(x,y,z) &= 3x+2y+z-1=0 \\ f_2(x,y,z) &= x+2y-3z-4=0 \\ L(x,y,z) &= \frac{1}{2}x^2+\frac{1}{2}y^2+\frac{1}{2}z^2 \end{aligned}$$

Let

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$\begin{aligned} H &= \frac{1}{2}w^T w + [\lambda_1 \quad \lambda_2] \begin{bmatrix} 3 & 2 & 1 & -1 \\ 1 & 2 & -3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \\ H_{\lambda_1} &= 3x+2y+z-1=0 \\ H_{\lambda_2} &= x+2y-3z+4=0 \\ H_x &= y+3\lambda_1+\lambda_2=0 \qquad (7) \\ H_y &= y+2\lambda_1+2\lambda_2=0 \qquad (8) \\ H_z &= z+\lambda_1-3\lambda_2=0 \qquad (9) \end{aligned}$$

From (7)-(9) we obtain

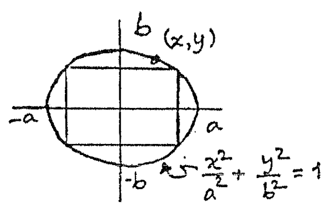
$$\begin{aligned} x &= -3\lambda_1-\lambda_2 \\ y &= -2\lambda_1-2\lambda_2 \\ z &= -\lambda_1+3\lambda_2 \end{aligned}$$

and

$$\begin{aligned} 7\lambda_1+2\lambda_2 &= -\frac{1}{2} \\ 2\lambda_1+7\lambda_2 &= -2 \end{aligned}$$

result in $\lambda_1 = \frac{1}{90}$, $\lambda_2 = -\frac{13}{95}$. Hence, $x = \frac{23}{90}$, $y = \frac{5}{9}$, $z = -\frac{79}{90}$.

1.2-10



a). Here we have,

$$
\begin{aligned}
L(x,y) &= 4(x+y) \\
f(x,y) &= \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \\
H(x,y) &= L+\lambda^T f = 4x+4y+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right) \\
H_x &= 4+\frac{2\lambda}{a^2}x=0 \qquad (11)\\
H_y &= 4+\frac{2\lambda}{b^2}y=0 \qquad (12)\\
H_\lambda &= \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \qquad (13)
\end{aligned}
$$

Equations (11)-(13) result in

$$
\begin{aligned}
\lambda &= -2\sqrt{a^2+b^2} \\
x &= \frac{a^2}{\sqrt{a^2+b^2}} \\
y &= \frac{b^2}{\sqrt{a^2+b^2}}
\end{aligned}
$$

$$
\begin{aligned}
L^f_{yy} &= H_{yy}-2\frac{f_y}{f_x}H_{xy}+\frac{f_y^2}{f_x^2}H_{xx} \\
&= 2\frac{\lambda}{b^2}+\frac{2\lambda y^2 a^2}{x^2 b^4}=\frac{2\lambda}{b^2}\left[1+\frac{a^2y^2}{b^2x^2}\right]<0, \quad iff \quad \lambda<0
\end{aligned}
$$

b).

$$
\begin{aligned}
L(x,y) &= 4xy, \quad f(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \\
H(x,y) &= L+\lambda^T f = 4xy+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right) \\
H_x &= 4y+\frac{2\lambda}{a^2}x=0 \qquad (14)\\
H_y &= 4+\frac{2\lambda}{b^2}y=0 \qquad (15)\\
H_\lambda &= \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \qquad (16)
\end{aligned}
$$

Equations (14)-(16) result in

$$
\lambda = -2ab,
$$

$$
\begin{aligned}
x &= \frac{a}{\sqrt{2}}, \\
y &= \frac{b}{\sqrt{2}}
\end{aligned}
$$

which gives the optimum area

$$L^{*} = \frac{ab}{2}.$$

This area is the maximum since

$$
\begin{aligned}
L^{J}_{yy} &= H_{yy} - 2\frac{f_y}{f_x}H_{xy} + \frac{f_y^2}{f_x^2}H_{xx} \\
&= -4\frac{a}{b}\left[1 + \frac{ay}{bx}\right]^2 < 0, \quad (a > 0, b > 0).
\end{aligned}
$$