Problem 1.1 Given the three vectors $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$, $\mathbf{B} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$ and $\mathbf{C} = C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}} + C_z\hat{\mathbf{k}}$, show analytically that

(a) $\mathbf{A}\cdot\mathbf{A} = A^2$
(b) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$
(c) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Solution

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{A} &= \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right)\left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right)\\
&= A_x\hat{\mathbf{i}}\cdot\left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right) + A_y\hat{\mathbf{j}}\cdot\left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right) + A_z\hat{\mathbf{k}}\cdot\left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right)\\
&= \left[A_x^2(\hat{\mathbf{i}}\cdot\hat{\mathbf{i}}) + A_xA_y(\hat{\mathbf{i}}\cdot\hat{\mathbf{j}}) + A_xA_z(\hat{\mathbf{i}}\cdot\hat{\mathbf{k}})\right] + \left[A_yA_x(\hat{\mathbf{j}}\cdot\hat{\mathbf{i}}) + A_y^2(\hat{\mathbf{j}}\cdot\hat{\mathbf{j}}) + A_yA_z(\hat{\mathbf{j}}\cdot\hat{\mathbf{k}})\right]\\
&\quad + \left[A_zA_x(\hat{\mathbf{k}}\cdot\hat{\mathbf{i}}) + A_zA_y(\hat{\mathbf{k}}\cdot\hat{\mathbf{j}}) + A_z^2(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}})\right]\\
&= \left[A_x^2(1) + A_xA_y(0) + A_xA_z(0)\right] + \left[A_yA_x(0) + A_y^2(1) + A_yA_z(0)\right] + \left[A_zA_x(0) + A_zA_y(0) + A_z^2(1)\right]\\
&= A_x^2 + A_y^2 + A_z^2
\end{aligned}$$

But, according to the Pythagorean Theorem, $A_x^2 + A_y^2 + A_z^2 = A^2$, where $A = \|\mathbf{A}\|$, the magnitude of the vector $\mathbf{A}$. Thus $\boxed{\mathbf{A}\cdot\mathbf{A} = A^2}$.

(b)

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \mathbf{A}\cdot\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}\\
&= \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right)\left[\hat{\mathbf{i}}\left(B_yC_z - B_zC_y\right) - \hat{\mathbf{j}}\left(B_xC_z - B_zC_x\right) + \hat{\mathbf{k}}\left(B_xC_y - B_yC_x\right)\right]\\
&= A_x\left(B_yC_z - B_zC_y\right) - A_y\left(B_xC_z - B_zC_x\right) + A_z\left(B_xC_y - B_yC_x\right)
\end{aligned}$$

or

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = A_xB_yC_z + A_yB_zC_x + A_zB_xC_y - A_xB_zC_y - A_yB_xC_z - A_zB_yC_x \qquad (1)$$

Note that $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$, and according to (1)

$$\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = C_xA_yB_z + C_yA_zB_x + C_zA_xB_y - C_xA_zB_y - C_yA_xB_z - C_zA_yB_x \qquad (2)$$

The right hand sides of (1) and (2) are identical. Hence $\boxed{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}}$.

(c)

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\left(A_x\hat{\mathbf{i}}+A_y\hat{\mathbf{j}}+A_z\hat{\mathbf{k}}\right)\times\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ A_x & A_y & A_z\\ B_yC_z-B_zC_y & B_zC_x-B_xC_z & B_xC_y-B_yC_x\end{vmatrix}$$

$$\begin{aligned}&=\left[A_y\left(B_xC_y-B_yC_x\right)-A_z\left(B_zC_x-B_xC_z\right)\right]\hat{\mathbf{i}}+\left[A_z\left(B_yC_z-B_zC_y\right)-A_x\left(B_xC_y-B_yC_x\right)\right]\hat{\mathbf{j}}\\&\quad+\left[A_x\left(B_zC_x-B_xC_z\right)-A_y\left(B_yC_z-B_zC_y\right)\right]\hat{\mathbf{k}}\\&=\left(A_yB_xC_y+A_zB_xC_z-A_yB_yC_x-A_zB_zC_x\right)\hat{\mathbf{i}}+\left(A_xB_yC_x+A_zB_yC_z-A_xB_xC_y-A_zB_zC_y\right)\hat{\mathbf{j}}\\&\quad+\left(A_xB_zC_x+A_yB_zC_y-A_xB_xC_z-A_yB_yC_z\right)\hat{\mathbf{k}}\\&=\left[B_x\left(A_yC_y+A_zC_z\right)-C_x\left(A_yB_y+A_zB_z\right)\right]\hat{\mathbf{i}}+\left[B_y\left(A_xC_x+A_zC_z\right)-C_y\left(A_xB_x+A_zB_z\right)\right]\hat{\mathbf{j}}\\&\quad+\left[B_z\left(A_xC_x+A_yC_y\right)-C_z\left(A_xB_x+A_yB_y\right)\right]\hat{\mathbf{k}}\end{aligned}$$

Add and subtract the underlined terms to get

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C})&=\left[B_x\left(A_yC_y+A_zC_z+\underline{A_xC_x}\right)-C_x\left(A_yB_y+A_zB_z+\underline{A_xB_x}\right)\right]\hat{\mathbf{i}}\\&\quad+\left[B_y\left(A_xC_x+A_zC_z+\underline{A_yC_y}\right)-C_y\left(A_xB_x+A_zB_z+\underline{A_yB_y}\right)\right]\hat{\mathbf{j}}\\&\quad+\left[B_z\left(A_xC_x+A_yC_y+\underline{A_zC_z}\right)-C_z\left(A_xB_x+A_yB_y+\underline{A_zB_z}\right)\right]\hat{\mathbf{k}}\\&=\left(B_x\hat{\mathbf{i}}+B_y\hat{\mathbf{j}}+B_z\hat{\mathbf{k}}\right)\left(A_xC_x+A_yC_y+A_zC_z\right)-\left(C_x\hat{\mathbf{i}}+C_y\hat{\mathbf{j}}+C_z\hat{\mathbf{k}}\right)\left(A_xB_x+A_yB_y+A_zB_z\right)\\&=\left(B_x\hat{\mathbf{i}}+B_y\hat{\mathbf{j}}+B_z\hat{\mathbf{k}}\right)(\mathbf{A}\cdot\mathbf{C})-\left(C_x\hat{\mathbf{i}}+C_y\hat{\mathbf{j}}+C_z\hat{\mathbf{k}}\right)(\mathbf{A}\cdot\mathbf{B})\end{aligned}$$

Or,

$$\boxed{\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})}$$

Problem 1.2 Use just the vector identities in Problem 1.1 to show that

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

Solution

From Problem 1.1(b)

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=\left[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}\right]\cdot\mathbf{D} \tag{1}$$

But

$$\left[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}\right]\cdot\mathbf{D}=-\left[\mathbf{C}\times(\mathbf{A}\times\mathbf{B})\right]\cdot\mathbf{D}$$

Using Problem 1.1(c) on the right yields

$$\left[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}\right]\cdot\mathbf{D}=-\left[\mathbf{A}(\mathbf{C}\cdot\mathbf{B})-\mathbf{B}(\mathbf{C}\cdot\mathbf{A})\right]\cdot\mathbf{D}$$

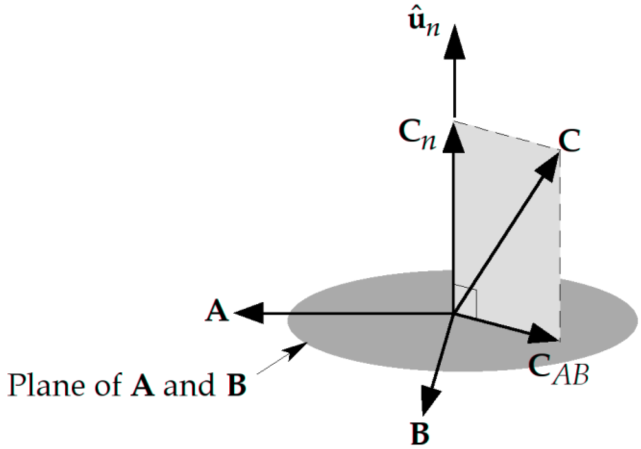
or

$$\left[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}\right]\cdot\mathbf{D}=-(\mathbf{A}\cdot\mathbf{D})(\mathbf{C}\cdot\mathbf{B})+(\mathbf{B}\cdot\mathbf{D})(\mathbf{C}\cdot\mathbf{A}) \tag{2}$$

Substituting (2) into (1) we get

$$\boxed{(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})}$$

Problem 1.3 Let $\mathbf{A} = 8\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$, $\mathbf{B} = 9\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{C} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 10\hat{\mathbf{k}}$. Calculate the (scalar) projection of C_{AB} of $\mathbf{C}$ onto the plane of $\mathbf{A}$ and $\mathbf{B}$.



Solution

The unit normal $\hat{\mathbf{u}}_n$ to the plane of **A** and **B** is

$$\hat{\mathbf{u}}_n = \frac{\mathbf{A}\times\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} = \frac{-63\hat{\mathbf{i}} + 100\hat{\mathbf{j}} - 33\hat{\mathbf{k}}}{17\cdot 10.863} = -0.34115\hat{\mathbf{i}} + 0.54141\hat{\mathbf{j}} - 0.17870\hat{\mathbf{k}}$$

The projection C_n of **C** in the direction of the unit normal to the plane is

$$C_n = \mathbf{C}\cdot\hat{\mathbf{u}}_n = \left(3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 10\hat{\mathbf{k}}\right)\left(-0.34115\hat{\mathbf{i}} + 0.54141\hat{\mathbf{j}} - 0.17870\hat{\mathbf{k}}\right) = -0.10289$$

C is the hypotenuse of the right triangle whose other two sides are of length C_n and C_{AB}, where C_{AB} lies in the plane of **A** and **B**. Therefore, the square of the length of **C** is

$$\|\mathbf{C}\|^2 = C_n{}^2 + C_{AB}{}^2$$

That is

$$3^2 + 5^2 + 10^2 = (-0.10289)^2 + C_{AB}{}^2$$

which means

$$\boxed{C_{AB} = 11.575}$$

Problem 1.4 Since $\hat{\mathbf{u}}_t$ and $\hat{\mathbf{u}}_n$ are perpendicular and $\hat{\mathbf{u}}_t \times \hat{\mathbf{u}}_n = \hat{\mathbf{u}}_b$, use the *bac-cab* rule to show that $\hat{\mathbf{u}}_b \times \hat{\mathbf{u}}_t = \hat{\mathbf{u}}_n$ and $\hat{\mathbf{u}}_n \times \hat{\mathbf{u}}_b = \hat{\mathbf{u}}_t$, thereby verifying Equation 1.29.

Solution

bac-cab rule: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

$$\begin{aligned}
\hat{\mathbf{u}}_b \times \hat{\mathbf{u}}_t &= (\hat{\mathbf{u}}_t \times \hat{\mathbf{u}}_n) \times \hat{\mathbf{u}}_t \\
&= -\hat{\mathbf{u}}_t \times (\hat{\mathbf{u}}_t \times \hat{\mathbf{u}}_n) \\
&= -\left[\hat{\mathbf{u}}_t (\hat{\mathbf{u}}_t \cdot \hat{\mathbf{u}}_n) - \hat{\mathbf{u}}_n (\hat{\mathbf{u}}_t \cdot \hat{\mathbf{u}}_t)\right] \\
&= -\left[\hat{\mathbf{u}}_t (0) - \hat{\mathbf{u}}_n (1)\right]
\end{aligned}$$

$\therefore \boxed{\hat{\mathbf{u}}_b \times \hat{\mathbf{u}}_t = \hat{\mathbf{u}}_n}$

$$\begin{aligned}
\hat{\mathbf{u}}_n \times \hat{\mathbf{u}}_b &= \hat{\mathbf{u}}_n \times (\hat{\mathbf{u}}_t \times \hat{\mathbf{u}}_n) \\
&= \hat{\mathbf{u}}_t (\hat{\mathbf{u}}_n \cdot \hat{\mathbf{u}}_n) - \hat{\mathbf{u}}_n (\hat{\mathbf{u}}_n \cdot \hat{\mathbf{u}}_t) \\
&= \hat{\mathbf{u}}_t (1) - \hat{\mathbf{u}}_n (0)
\end{aligned}$$

$\therefore \boxed{\hat{\mathbf{u}}_n \times \hat{\mathbf{u}}_b = \hat{\mathbf{u}}_t}$

Problem 1.5 The *x*, *y* and *z* coordinates (in meters) of a particle as a function of time (in seconds) are $x = \sin 3t$, $y = \cos t$ and $z = \sin 2t$. At $t = 3\text{s}$ determine:

(a) The velocity **v**, in Cartesian coordinates.
(b) The speed *v*.
(c) The unit tangent $\hat{\mathbf{u}}_t$.
(d) The angles θ_x, θ_y and θ_z that **v** makes with the *x*, *y* and *z* axes.
(e) The acceleration **a** in Cartesian coordinates.
(f) The unit binormal vector $\hat{\mathbf{u}}_b$.
(g) The unit normal vector $\hat{\mathbf{u}}_n$.
(h) The angles ϕ_x, ϕ_y and ϕ_z that **a** makes with the *x*, *y* and *z* axes.
(i) The tangential component a_t of the acceleration.
(j) The normal component a_n of the acceleration.
(k) The radius of curvature of the path of *P*.
(l) The Cartesian coordinates of the center of curvature of the path.

Solution

(a)

$$\mathbf{v} = \left.\frac{d\mathbf{r}}{dt}\right)_{t=3} = \frac{d}{dt}\left(\sin 3t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}} + \sin 2t\hat{\mathbf{k}}\right)\Big]_{t=3} = 3\cos 3t\hat{\mathbf{i}} - \sin t\hat{\mathbf{j}} + 2\cos 2t\hat{\mathbf{k}}\Big)_{t=3}$$

$$\boxed{\mathbf{v} = -2.2791\hat{\mathbf{i}} + 0.95892\hat{\mathbf{j}} - 1.6781\hat{\mathbf{k}}\ (\text{m/s})}$$

(b)

$$v = \|\mathbf{v}\| = \sqrt{(-2.2791)^2 + 0.95892^2 + (-1.6781)^2}$$

$$\boxed{v = 2.9883\ \text{m/s}}$$

(c)

$$\hat{\mathbf{u}}_t = \frac{\mathbf{v}}{v} = \frac{-2.2791\hat{\mathbf{i}} + 0.95892\hat{\mathbf{j}} - 1.6781\hat{\mathbf{k}}}{2.9883}$$

$$\boxed{\hat{\mathbf{u}}_t = -0.76267\hat{\mathbf{i}} + 0.32089\hat{\mathbf{j}} - 0.56157\hat{\mathbf{k}}}$$

(d)

$$\theta_x = \cos^{-1}\left(\hat{\mathbf{u}}_t \cdot \hat{\mathbf{i}}\right) = \cos^{-1}(-0.76267)$$

$$\boxed{\theta_x = 139.70°}$$

$$\theta_y = \cos^{-1}\left(\hat{\mathbf{u}}_t \cdot \hat{\mathbf{j}}\right) = \cos^{-1}(0.32089)$$

$$\boxed{\theta_y = 71.283°}$$

$$\theta_z = \cos^{-1}\left(\hat{\mathbf{u}}_t \cdot \hat{\mathbf{k}}\right) = \cos^{-1}(-0.56157)$$

$$\boxed{\theta_z = 124.16°}$$

(e)

$$\mathbf{a} = \left.\frac{d\mathbf{v}}{dt}\right)_{t=3} = \frac{d}{dt}\left(3\cos 3t\hat{\mathbf{i}} - \sin t\hat{\mathbf{j}} + 2\cos 2t\hat{\mathbf{k}}\right)\Big]_{t=3} = -9\sin 3t\hat{\mathbf{i}} - \cos t\hat{\mathbf{j}} - 4\sin 2t\hat{\mathbf{k}}\Big)_{t=3}$$

$$\boxed{\mathbf{a} = -5.8526\hat{\mathbf{i}} - 0.28366\hat{\mathbf{j}} + 2.1761\hat{\mathbf{k}}\ \left(\text{m/s}^2\right)}$$

(f)

$$\hat{\mathbf{u}}_b = \frac{\mathbf{v}\times\mathbf{a}}{\|\mathbf{v}\times\mathbf{a}\|} = \frac{\left(-2.2791\hat{\mathbf{i}} + 0.95892\hat{\mathbf{j}} - 1.6781\hat{\mathbf{k}}\right)\times\left(-5.8526\hat{\mathbf{i}} - 0.28366\hat{\mathbf{j}} + 2.1761\hat{\mathbf{k}}\right)}{\left\|\left(-2.2791\hat{\mathbf{i}} + 0.95892\hat{\mathbf{j}} - 1.6781\hat{\mathbf{k}}\right)\times\left(-5.8526\hat{\mathbf{i}} - 0.28366\hat{\mathbf{j}} + 2.1761\hat{\mathbf{k}}\right)\right\|}$$

$$= \frac{1.6107\hat{\mathbf{i}} + 14.781\hat{\mathbf{j}} + 6.2587\hat{\mathbf{k}}}{\left\|1.6107\hat{\mathbf{i}} + 14.781\hat{\mathbf{j}} + 6.2587\hat{\mathbf{k}}\right\|} = \frac{1.6107\hat{\mathbf{i}} + 14.781\hat{\mathbf{j}} + 6.2587\hat{\mathbf{k}}}{16.132}$$

$$\boxed{\hat{\mathbf{u}}_b = 0.099844\hat{\mathbf{i}} + 0.91625\hat{\mathbf{j}} + 0.38797\hat{\mathbf{k}}}$$

(g)

$$\hat{\mathbf{u}}_n = \hat{\mathbf{u}}_b \times \hat{\mathbf{u}}_t = \left(0.099844\hat{\mathbf{i}} + 0.91625\hat{\mathbf{j}} + 0.38797\hat{\mathbf{k}}\right)\times\left(-0.76267\hat{\mathbf{i}} + 0.32089\hat{\mathbf{j}} - 0.56157\hat{\mathbf{k}}\right)$$

$$\boxed{\hat{\mathbf{u}}_n = -0.63904\hat{\mathbf{i}} - 0.23982\hat{\mathbf{j}} + 0.73083\hat{\mathbf{k}}}$$

(h)

$$a = \|\mathbf{a}\| = \sqrt{(-5.8526)^2 + (-0.28366)^2 + (2.1761)^2} = 6.2505\ \text{m/s}^2$$

$$\phi_x = \cos^{-1}\left(\frac{a_x}{a}\right) = \cos^{-1}\left(\frac{-5.8526}{6.2505}\right)$$

$$\boxed{\phi_x = 159.45^\circ}$$

$$\phi_y = \cos^{-1}\left(\frac{a_y}{a}\right) = \cos^{-1}\left(\frac{-0.28366}{6.2505}\right)$$

$$\boxed{\phi_y = 92.601^\circ}$$

$$\phi_z = \cos^{-1}\left(\frac{a_z}{a}\right) = \cos^{-1}\left(\frac{2.1761}{6.2505}\right)$$

$$\boxed{\phi_z = 69.626^\circ}$$

(i)

$$a_t = \mathbf{a}\cdot\hat{\mathbf{u}}_t = \left(-5.8526\hat{\mathbf{i}} - 0.28366\hat{\mathbf{j}} + 2.1761\hat{\mathbf{k}}\right)\left(-0.76267\hat{\mathbf{i}} + 0.32089\hat{\mathbf{j}} - 0.56157\hat{\mathbf{k}}\right)$$

$$\boxed{a_t = 3.1505\ \text{m/s}^2}$$

(j)

$$a_n = \mathbf{a}\cdot\hat{\mathbf{u}}_n = \left(-5.8526\hat{\mathbf{i}} - 0.28366\hat{\mathbf{j}} + 2.1761\hat{\mathbf{k}}\right)\left(-0.63904\hat{\mathbf{i}} - 0.23982\hat{\mathbf{j}} + 0.73083\hat{\mathbf{k}}\right)$$

$$\boxed{a_n = 5.3984\ \text{m/s}^2}$$

(k)

$$\rho = \frac{v^2}{a_n} = \frac{2.9883^2}{5.3984}$$

$$\boxed{\rho = 1.6542\ \text{m}}$$

(l)

$$\mathbf{r}_C = \mathbf{r} + \rho\hat{\mathbf{u}}_n$$

$$= \left(0.65029\hat{\mathbf{i}} + 0.28366\hat{\mathbf{j}} - 0.54402\hat{\mathbf{k}}\right) + 1.6542\left(-0.63904\hat{\mathbf{i}} - 0.23982\hat{\mathbf{j}} + 0.73083\hat{\mathbf{k}}\right)$$

$$\boxed{\mathbf{r}_C = -0.40678\hat{\mathbf{i}} - 0.11304\hat{\mathbf{j}} + 0.66489\hat{\mathbf{k}}\ (\text{m})}$$

Problem 1.6 An 80 kg man and 50 kg woman stand 0.5 meter from each other. What is the force of gravitational attraction between the couple?

Solution

$$F = \frac{Gm_1m_2}{r^2} = \frac{6.6742 \times 10^{-11} \cdot 80 \cdot 50}{0.5^2}$$

$$\boxed{F = 1.0679 \times 10^{-6}\ \text{N}}$$

Problem 1.7 If a person's weight is W on the surface of the earth, calculate the earth's gravitational pull on that person at a distance equal to the moon's orbit.

Solution

Let m be the person's mass, M_E the mass of the earth, and R_E the radius of the earth. Then

$$W = \frac{GmM_E}{{R_E}^2}$$

or

$$GmM_E = W{R_E}^2 \tag{1}$$

If r_{moon} is the moon's orbital radius, then the gravitational pull of the earth at that distance is

$$F = \frac{GmM_E}{{r_{\text{moon}}}^2} \tag{2}$$

Substituting (1) into (2) yields

$$F = \left(\frac{R_E}{r_{\text{moon}}}\right)^2 W$$

Since $R_E = 6378\text{ km}$ and $r_{\text{moon}} = 384\,400\text{ km}$, we find

$$\boxed{F = 0.0002753W}$$

Problem 1.8 If a person's weight is *W* on the surface of the earth, calculate what it would be, in terms of *W*, at the surface of

(a) the moon

(b) Mars

(c) Jupiter

Solution

The force of gravity on mass *m* at the earth's surface is

$$W = \frac{GmM_E}{{R_E}^2}$$

so that

$$Gm = \frac{W{R_E}^2}{M_E} \tag{1}$$

At the surface of planet *P*, the force of gravity is. using (1),

$$W_P = \frac{GmM_P}{{R_P}^2} = \left(\frac{R_E}{R_P}\right)^2 \frac{M_P}{M_E} W \tag{2}$$

(a)

$$W_{\text{moon}} = \left(\frac{R_E}{R_{\text{moon}}}\right)^2 \frac{M_{\text{moon}}}{M_E} W = \left(\frac{6378}{1737}\right)^2 \frac{7.348 \times 10^{22}}{597.4 \times 10^{22}} W$$

$$\boxed{W_{\text{moon}} = 0.1658W}$$

(b)

$$W_{\text{Mars}} = \left(\frac{R_E}{R_{\text{Mars}}}\right)^2 \frac{M_{\text{Mars}}}{M_E} W = \left(\frac{6378}{3396}\right)^2 \frac{64.19 \times 10^{22}}{597.4 \times 10^{22}} W$$

$$\boxed{W_{\text{Mars}} = 0.3790W}$$

(c)

$$W_{\text{Jupiter}} = \left(\frac{R_E}{R_{\text{Jupiter}}}\right)^2 \frac{M_{\text{Jupiter}}}{M_E} W = \left(\frac{6378}{71\,490}\right)^2 \frac{189\,900 \times 10^{22}}{597.4 \times 10^{22}} W$$

$$\boxed{W_{\text{Jupiter}} = 2.530W}$$

Problem 1.9 A satellite of mass m is in a circular orbit around the earth, whose mass is M. The orbital radius from the center of the earth is r. Use Newton's Second Law of motion, together with Equations 1.25 and 1.40, to calculate the speed v of the satellite in terns of M, r and the gravitational constant G.

Solution

Writing Newton's Second Law in the direction normal to the circular orbital path,

$$F_n = ma_n \tag{1}$$

From Equation 1.25

$$a_n = \frac{v^2}{r}$$

From Equation 1,40m

$$F_n = \frac{GmM}{r^2}$$

Therefore, (1) becomes

$$\frac{mv^2}{r} = \frac{GmM}{r^2}$$

so that

$$\boxed{v = \sqrt{\frac{GM}{r}}}$$

Problem 1.10 If the earth takes 365.25 days to complete its circular orbit of radius 149.6×10^6 km around the sun, use the result of Problem 1.9 to calculate the mass of the sun.

Solution

From Problem 1.9

$$v = \sqrt{\frac{GM}{r}}$$

The constant speed v equals the distance $2\pi r$ traveled in one period T,

$$v = \frac{2\pi r}{T}$$

Thus

$$\sqrt{\frac{GM}{r}} = \frac{2\pi r}{T}$$

or

$$M = \frac{4\pi^2}{G}\frac{r^3}{T^2}$$

Thus

$$M = \frac{4\pi^2}{6.6742 \times 10^{-20}} \frac{\left(1.496 \times 10^8\right)^3}{\left(365.25 \cdot 24 \cdot 3600\right)^2}$$

$$\boxed{M = 1.9886 \times 10^{30} \text{ kg}}$$

Problem 1.11 $\mathbf{F}$ is a force vector of fixed magnitude embedded on a rigid body in plane motion (in the *xy* plane). At a given instant $\boldsymbol{\omega} = 2\hat{\mathbf{k}}\ \text{rad/s}$, $\dot{\boldsymbol{\omega}} = -5\hat{\mathbf{k}}\ \text{rad/s}^2$, $\ddot{\boldsymbol{\omega}} = 3\hat{\mathbf{k}}\ \text{rad/s}^3$ and $\mathbf{F} = 15\hat{\mathbf{i}} + 10\hat{\mathbf{j}}\ \text{N}$. At that instant calculate $\dddot{\mathbf{F}}$.

Solution

From Example 1.12, we have

$$\dddot{\mathbf{F}} = \ddot{\boldsymbol{\omega}} \times \mathbf{F} + 2\dot{\boldsymbol{\omega}} \times (\boldsymbol{\omega} \times \mathbf{F}) + \boldsymbol{\omega} \times \left[\dot{\boldsymbol{\omega}} \times \mathbf{F} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{F})\right]$$

Substituting the given values for the quantities on the right hand side,

$$\ddot{\boldsymbol{\omega}} \times \mathbf{F} = 3\hat{\mathbf{k}} \times (15\hat{\mathbf{i}} + 10\hat{\mathbf{j}}) = -30\hat{\mathbf{i}} + 45\hat{\mathbf{j}}\ \left(\text{N/s}^3\right)$$

$$2\dot{\boldsymbol{\omega}} \times (\boldsymbol{\omega} \times \mathbf{F}) = 2(-5\hat{\mathbf{k}}) \times \left[(2\hat{\mathbf{k}}) \times (15\hat{\mathbf{i}} + 10\hat{\mathbf{j}})\right] = 2(-5\hat{\mathbf{k}}) \times (-20\hat{\mathbf{i}} + 30\hat{\mathbf{j}}) = 2(150\hat{\mathbf{i}} + 100\hat{\mathbf{j}}) = 300\hat{\mathbf{i}} + 200\hat{\mathbf{j}}\left(\text{N/s}^3\right)$$

$$\boldsymbol{\omega} \times (\dot{\boldsymbol{\omega}} \times \mathbf{F}) = (2\hat{\mathbf{k}}) \times \left[(-5\hat{\mathbf{k}}) \times (15\hat{\mathbf{i}} + 10\hat{\mathbf{j}})\right] = (2\hat{\mathbf{k}}) \times (50\hat{\mathbf{i}} - 75\hat{\mathbf{j}}) = 150\hat{\mathbf{i}} + 100\hat{\mathbf{j}}\ \left(\text{N/s}^3\right)$$

$$\begin{aligned}\boldsymbol{\omega} \times \left[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{F})\right] &= (2\hat{\mathbf{k}}) \times \left\{(2\hat{\mathbf{k}}) \times \left[(2\hat{\mathbf{k}}) \times (15\hat{\mathbf{i}} + 10\hat{\mathbf{j}})\right]\right\} = (2\hat{\mathbf{k}}) \times \left[(2\hat{\mathbf{k}}) \times (-20\hat{\mathbf{i}} + 30\hat{\mathbf{j}})\right] = (2\hat{\mathbf{k}}) \times (-60\hat{\mathbf{i}} - 40\hat{\mathbf{j}}) \\ &= 80\hat{\mathbf{i}} - 120\hat{\mathbf{j}}\ \left(\text{N/s}^3\right)\end{aligned}$$

Thus,

$$\dddot{\mathbf{F}} = (-30\hat{\mathbf{i}} + 45\hat{\mathbf{j}}) + (300\hat{\mathbf{i}} + 200\hat{\mathbf{j}}) + (150\hat{\mathbf{i}} + 100\hat{\mathbf{j}}) + (80\hat{\mathbf{i}} - 120\hat{\mathbf{j}})$$

$$\boxed{\dddot{\mathbf{F}} = 500\hat{\mathbf{i}} + 225\hat{\mathbf{j}}\ \left(\text{N/s}^3\right)}$$

Problem 1.12 The absolute position, velocity and acceleration of O are

$$\mathbf{r}_O = -16\hat{\mathbf{I}} + 84\hat{\mathbf{J}} + 59\hat{\mathbf{K}} \text{ (m)}$$
$$\mathbf{v}_O = 7\hat{\mathbf{I}} + 9\hat{\mathbf{J}} + 4\hat{\mathbf{K}} \text{ (m/s)}$$
$$\mathbf{a}_O = 3\hat{\mathbf{I}} - 7\hat{\mathbf{J}} + 4\hat{\mathbf{K}} \text{ (m/s}^2\text{)}$$

The angular velocity and acceleration of the moving frame are

$$\boldsymbol{\Omega} = -0.8\hat{\mathbf{I}} + 0.7\hat{\mathbf{J}} + 0.4\hat{\mathbf{K}} \text{ (rad/s)} \qquad \dot{\boldsymbol{\Omega}} = -0.4\hat{\mathbf{I}} + 0.9\hat{\mathbf{J}} - 1.0\hat{\mathbf{K}} \text{ (rad/s}^2\text{)}$$

The unit vectors of the moving frame are

$$\hat{\mathbf{i}} = -0.15617\hat{\mathbf{I}} - 0.31235\hat{\mathbf{J}} + 0.93704\hat{\mathbf{K}}$$
$$\hat{\mathbf{j}} = -0.12940\mathbf{I} + 0.94698\hat{\mathbf{J}} + 0.29409\hat{\mathbf{K}}$$
$$\hat{\mathbf{k}} = -0.97922\mathbf{I} - 0.075324\hat{\mathbf{J}} - 0.18831\hat{\mathbf{K}}$$

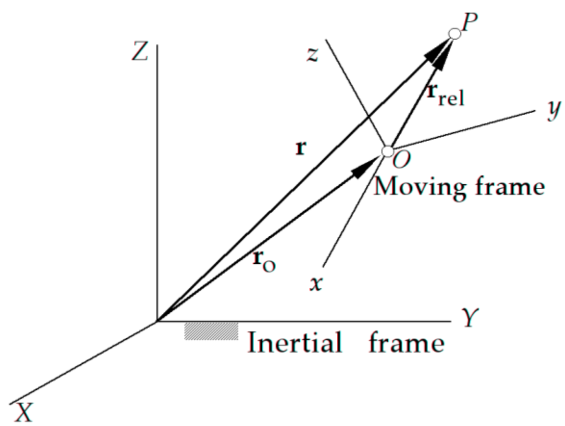
The absolute position of P is

$$\mathbf{r} = 51\hat{\mathbf{I}} - 45\hat{\mathbf{J}} + 36\hat{\mathbf{K}} \text{ (m)}$$

The velocity and acceleration of P relative to the moving frame are

$$\mathbf{v}_{\text{rel}} = 31\hat{\mathbf{i}} - 68\hat{\mathbf{j}} - 77\hat{\mathbf{k}} \text{ (m/s)} \qquad \mathbf{a}_{\text{rel}} = 2\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 5\hat{\mathbf{k}} \text{ (m/s)}$$

Calculate the absolute velocity $\mathbf{v}_P$ and acceleration $\mathbf{a}_P$ of P.



Solution

Velocity analysis

From Equation 1.66,

$$\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\Omega} \times \mathbf{r}_{\text{rel}} + \mathbf{v}_{\text{rel}}. \tag{1}$$