Exercise 2.1.

Define $ksi := 10^3 psi$ $\mu\varepsilon := 10^{-6}$ in/in

At a point in a thin plate the Cartesian stresses are $\sigma_x := 60ksi$, $\sigma_y := 6ksi$ and $\tau_{xy} := 13ksi$. If the plate is made of aluminum ($E := 10 \times 10^6 psi$, $\nu := \frac{1}{3}$). Determine the components of the strain tensor, ε_{ij}. In the stress free state the thickness of the plate, $t := 0.25000in$. What is the thickness at the point for the stress state given?

Solution

$$\sigma_{ij} := \begin{pmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad G := \frac{E}{2(1+\nu)}$$

The components of the strain tensor are:

$$\varepsilon_x := \frac{1}{E}(\sigma_x - \nu \cdot \sigma_y) \quad \varepsilon_y := \frac{1}{E}(\sigma_y - \nu \cdot \sigma_x) \quad \varepsilon_z := \frac{-\nu}{E}(\sigma_x + \sigma_y) \quad \gamma_{xy} := \frac{\tau_{xy}}{G}$$

The results are:

$$\varepsilon_x = 5800\,\mu\varepsilon \qquad \varepsilon_y = -1400\,\mu\varepsilon \qquad \varepsilon_z = -2200\,\mu\varepsilon \qquad \gamma_{xy} = 3467\,\mu\varepsilon$$

The thickness at the point is given by: $t' := t + t \cdot \varepsilon_z$

Therefore: $t' = 0.24945\,in$

Exercise 2.2

Given the Airy stress function, $F(x,y,d,P) := -\frac{P}{d^3}(x\cdot y)^2(3d-2y)$

(a) determine the corresponding Cartesian components of stress;
(b) what problem is solved by this stress function over the region bounded by the lines y=0 y=d, x=0 on the side x positive?

Solution:

Recall $\sigma_x(x,y,d,P) := \frac{d^2}{dy^2}\left[-\frac{P}{d^3}(x\cdot y)^2(3d-2y)\right] \rightarrow -2\cdot\frac{P}{d^3}\cdot x^2\cdot(3\cdot d-2\cdot y)+8\cdot\frac{P}{d^3}\cdot x^2\cdot y$

$$\sigma_y(x,y,d,P) := \frac{d^2}{dx^2}\left[-\left(\frac{P}{d^3}\right)(x\cdot y)^2(3d-2y)\right] \rightarrow -2\cdot\frac{P}{d^3}\cdot y^2\cdot(3\cdot d-2\cdot y)$$

$$\tau_{xy}(x,y,d,P) := -1\cdot\left[\frac{d}{dx}\left[-\frac{P}{d^3}\cdot(x\cdot y)^2\cdot(3\cdot d-2\cdot y)\right]\cdot\frac{d}{dy}\left[-\frac{P}{d^3}\cdot(x\cdot y)^2\cdot(3\cdot d-2\cdot y)\right]\right].$$

$$\tau_{xy}(x,y,d,P) := -24\cdot P^2\cdot x^3\cdot y^3\cdot(-3\cdot d+2\cdot y)\cdot\frac{(-d+y)}{d^6}$$

Continued differentiation reveals that the given function satisfies the biharmonic equation, and thus is a valid stress function.

On the edge x=0, the boundary conditions are:

$\sigma_x(0,y,d,P) \rightarrow 0 \qquad \tau_{xy}(0,y,d,P) \rightarrow 0$

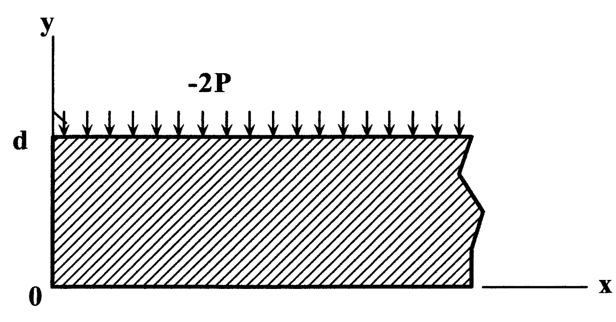
On the edge y=0, the boundary conditions are:

$\sigma_y(x,0,d,P) \rightarrow 0 \quad \tau_{xy}(x,0,d,P) \rightarrow 0$

On the edge y=d, the boundary conditions are:

$\sigma_y(x,d,d,P) \rightarrow -2\cdot P \qquad \tau_{xy}(x,d,d,P) \rightarrow 0$

Therefore the problem solved by the given Airy stress functions is that of a cantilever beam with a uniforml distributed force on the top edge (y=d) of magnitude -2P.



Exercise 2.3
Using the polar form of the strain definitions Eqs. (2.10) and (2.11), verify Eq. (2.38).

Solution:
For plane strain the non-zero strain components in polar form are:

$$\varepsilon_r = \frac{\partial u_r}{\partial r} \qquad \varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \qquad \gamma_{r\theta} = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}$$

Then

$$\frac{\partial \gamma_{r\theta}}{\partial \theta} = \frac{\partial^2 u_\theta}{\partial \theta \partial r} + \frac{1}{r}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \qquad (1)$$

and

$$\frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} = \frac{\partial^3 u_\theta}{\partial r \partial \theta \partial r} - \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{1}{r}\frac{\partial^3 u_r}{\partial r \partial \theta^2} + \frac{1}{r}\frac{\partial^3 u_r}{\partial r \partial \theta^2} + \frac{1}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{1}{r}\frac{\partial^2 u_\theta}{\partial r \partial \theta} \qquad (2)$$

but, from (1)

$$-\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} = -\frac{1}{r}\frac{\partial \gamma_{r\theta}}{\partial \theta} + \frac{1}{r}\frac{\partial^2 u_\theta}{\partial \theta \partial r} - \frac{1}{r^2}\frac{\partial u_\theta}{\partial \theta} \qquad (3)$$

Substituting (3) into (2) and noting that the order of differentiation can be interchanged IF the strains are continuous (i.e., the condition for compatibility) yields

$$\frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} + \frac{1}{r}\frac{\partial \gamma_{r\theta}}{\partial \theta} = \frac{\partial^3 u_\theta}{\partial r^2 \partial \theta} + \frac{1}{r}\frac{\partial^3 u_r}{\partial r \partial \theta^2}$$

$$= \frac{\partial^2}{\partial r^2}(r\varepsilon_\theta - u_r) + \frac{1}{r}\frac{\partial^2 \varepsilon_r}{\partial \theta^2}$$

$$= 2\frac{\partial \varepsilon_\theta}{\partial r} + r\frac{\partial^2 \varepsilon_\theta}{\partial r^2} - \frac{\partial \varepsilon_r}{\partial r} + \frac{1}{r}\frac{\partial^2 \varepsilon_r}{\partial \theta^2}$$

Finally, multiplying by r results in Eq. (2.38):

$$\frac{1}{r}\frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \gamma_{r\theta}}{\partial \theta} = \frac{2}{r}\frac{\partial \varepsilon_\theta}{\partial r} + \frac{\partial^2 \varepsilon_\theta}{\partial r^2} - \frac{1}{r}\frac{\partial \varepsilon_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varepsilon_r}{\partial \theta^2}$$

Exercise 2.4
Using the polar definition of the Airy stress function, Eq. (2.39), derive the polar form of the biharmonic equation from the compatibility equation (2.38)

Solution:
The compatibility equation in polar form is:

$$\frac{1}{r}\frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \gamma_{r\theta}}{\partial \theta} = \frac{2}{r}\frac{\partial \varepsilon_\theta}{\partial r} + \frac{\partial^2 \varepsilon_\theta}{\partial r^2} - \frac{1}{r}\frac{\partial \varepsilon_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varepsilon_r}{\partial \theta^2} \qquad (1)$$

The constitutive equations in plane strain are:

$$\varepsilon_r = \frac{1+\nu}{E}[\sigma_r - \nu(\sigma_r + \sigma_\theta)]$$

$$\varepsilon_\theta = \frac{1+\nu}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_\theta)] \qquad (2)$$

$$\gamma_{r\theta} = \frac{2(1+\nu)}{E}\tau_{r\theta}$$

Substituting Eqns. (2) into Eq. (1) and collecting terms, we find that:

$$\frac{2}{r}\frac{\partial^2 \tau_{r\theta}}{\partial r \partial \theta} + \frac{2}{r^2}\frac{\partial \tau_{r\theta}}{\partial \theta} = \frac{\partial^2 \sigma_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \sigma_r}{\partial \theta^2} + \frac{2}{r}\frac{\partial \sigma_\theta}{\partial r} - \frac{1}{r}\frac{\partial \sigma_r}{\partial r} - \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\sigma_r + \sigma_\theta) \qquad (4)$$

The equilibrium equations in polar form are:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (5a)$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_r}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0 \qquad (5b)$$

Multiply Eq. (5a) by *r* and differentiate with respect to *r* and differentiate Eq. (5b) with respect to *θ*, then substitute the results for the left hand side of Eq. (4). Collecting terms yields:

$$(1-\nu)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\sigma_r + \sigma_\theta) = 0 \qquad (6)$$

But, from Airy's definitions in polar coordinates:

$$\sigma_r = \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} \quad \text{and} \quad \sigma_\theta = \frac{\partial^2 \psi}{\partial r^2}$$

$$\sigma_r + \sigma_\theta = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = \nabla^2 \psi$$

Finally: Eq. (6) becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\psi = 0 \quad \text{or} \quad \nabla^2\nabla^2\psi = 0$$

Exercise 2.5

The general form of the Airy stress function for radially symmetric problems is of the form, $\mathcal{F} = A \log r + Br^2 \log r + Cr^2 + D$. Derive expressions for the polar stress components, σ_r, σ_θ for a large plate with a central hole of radius, *a,* which is subjected to a uniform radial stress, σ_o at a large distance from the hole.

Solution:

From radial symmetry the stresses reduce to;

$$\sigma_r = \frac{1}{r}\frac{dF}{dr} = \frac{A}{r^2} + B(1 + 2\log r) + 2C$$

$$\sigma_\theta = \frac{d^2F}{dr^2} = -\frac{A}{r^2} + B(3 + 2\log r) + 2C$$

If the stresses are to remain finite as $r \rightarrow \infty$, then $B=0$.

Imposing the boundary conditions that $\sigma_r = 0$ on $r=a$ and $\sigma_r = \sigma_o$ as $r \rightarrow \infty$, reduces the above equations to:

$$\sigma_r = \sigma_o(1 - \frac{a^2}{r^2}) \text{ and } \sigma_\theta = \sigma_o(1 + \frac{a^2}{r^2})$$

Exercise 2.6

Plot the distribution of $\sigma_{\beta\beta}$ along the major axis of an ellipse in biaxial tension for *a/b* = 2, 5,10. What conclusions can you draw from these results?

Solution:

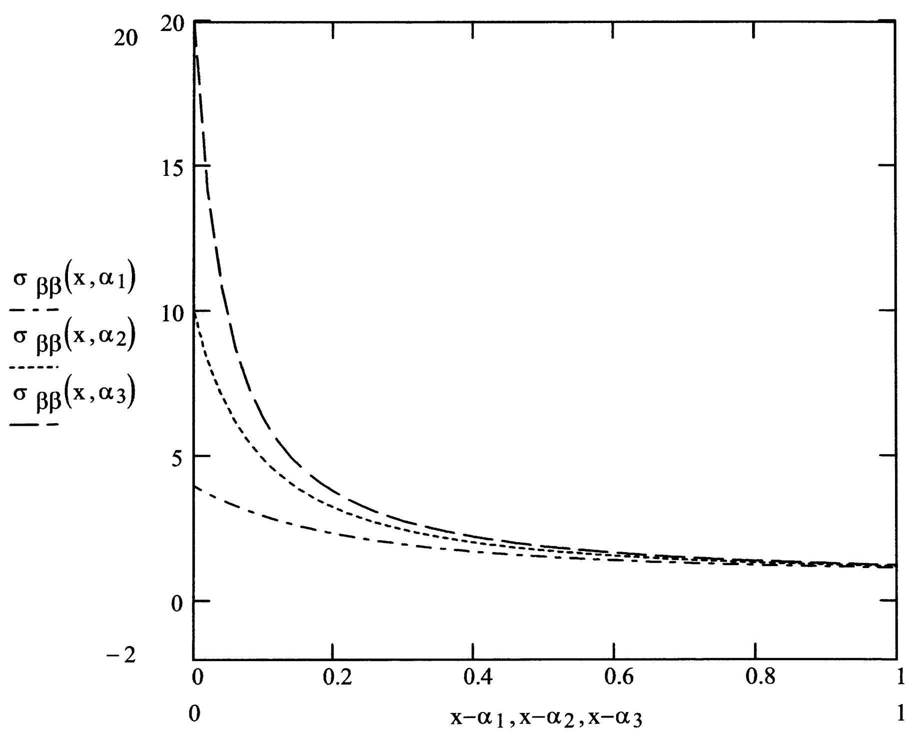
Since a/b=coth(α_0) For $i := 1..3$

Let $\alpha_1 := \text{acoth}(2)$ $\alpha_2 := \text{acoth}(5)$ $\alpha_3 := \text{acoth}(10)$

Then, for a unit applied load:

$$\sigma_{\beta\beta}(x,\alpha) := \left[\frac{\sinh[2(x)][\cosh[2(x)] + \cosh(2\alpha) - 2]}{[\cosh[2(x)] - 1]^2} \right]$$

and the distribution of $\sigma_{\beta\beta}$ along the major axis is



Conclusion: The stress gradient rises sharply as the ellipticity increases but the stress decays rapidly in all cases.

Exercise 2.7

Plot the distribution of $\sigma_{\beta\beta}$ around the boundary of an elliptical hole for $a/b = 2, 5, 10$ for equal biaxial remote tension.

Solution:

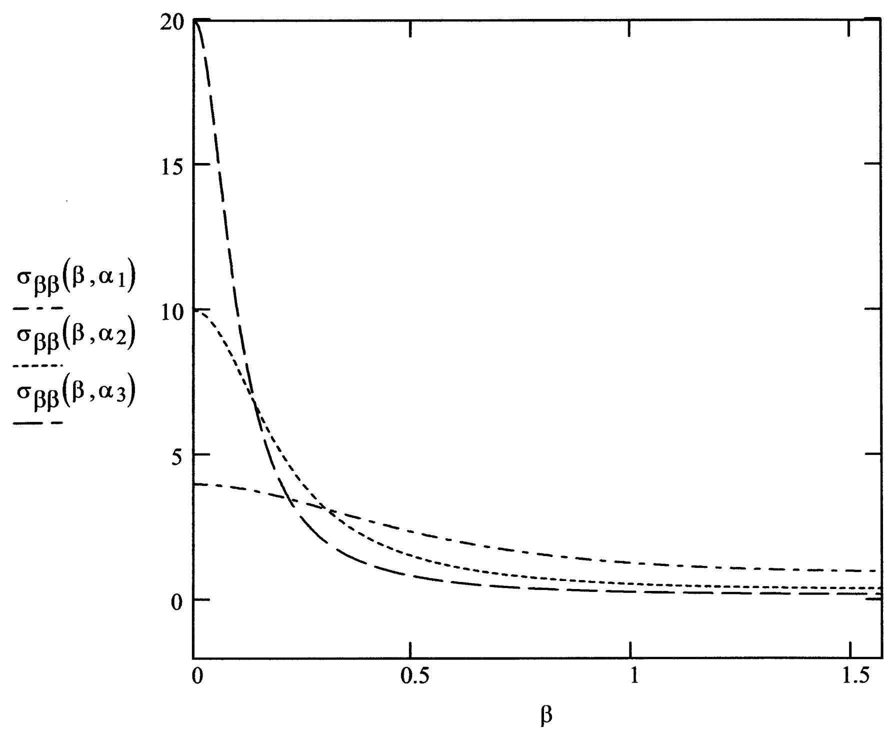
Since $a/b=\coth(\alpha_0)$ For $i := 1..3$

Let $\alpha_1 := \text{acoth}(2)$ $\alpha_2 := \text{acoth}(5)$ $\alpha_3 := \text{acoth}(10)$

Then, for a unit applied load:

$$\sigma_{\beta\beta}(\beta,\alpha) := \left[\frac{2\sinh(2\alpha)(\cosh(2\alpha)-\cos(2\beta))}{(\cosh(2\alpha)-\cos(2\beta))^2}\right]$$

and the distribution of $\sigma_{\beta\beta}$ around the elliptical boundary is



Exercise 2.8

Plot the distribution of $\sigma_{\beta\beta}$ around the boundary of an elliptical hole for *a/b* = 2, 5, 10 for uniaxial remote tension perpendicular to the major axis of the ellipse.

Solution:

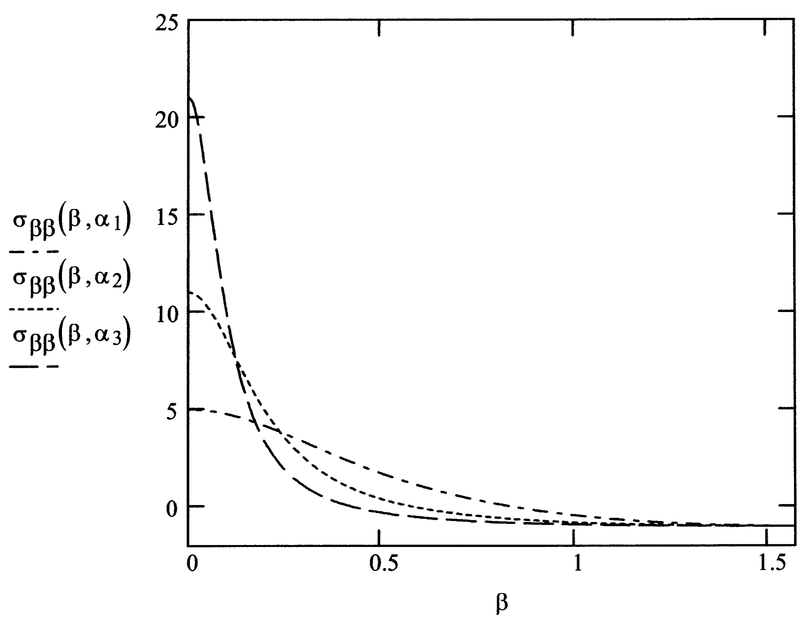
Since $a/b=\coth(\alpha_0)$ For $i := 1..3$

Let $\alpha_1 := \mathrm{acoth}(2)$ $\alpha_2 := \mathrm{acoth}(5)$ $\alpha_3 := \mathrm{acoth}(10)$

Then, for a unit applied load:

$$\sigma_{\beta\beta}(\beta,\alpha) := e^{2\alpha}\left[\frac{\left(1+e^{-2\alpha}\right)\sinh(2\alpha)}{\cosh(2\alpha)-\cos(2\beta)} - 1\right]$$

and the distribution of tangential stress around the ellipse boundary is:



Exercise 2.9
Show that Eq. (2.65) follows from Eq. (2.64).

Solution:
Since the maximum stress occurs along the major axis of the ellipse, from Eq. (2.64):

$$SCF = \sigma_{\beta\beta}\Big|_{\substack{\alpha=\alpha_o \\ \beta=0}} / \sigma = e^{2\alpha_o}\left[\frac{(1+e^{-2\alpha_o})\sinh 2\alpha_o}{\cosh 2\alpha_o - 1} - 1\right]$$

Expanding out and noting the following identities:
$cosh 2\alpha_o - \sinh 2\alpha_o = e^{-2\alpha_o}$ and $e^{2\alpha_o} - 1 = \sinh 2\alpha_o + \cosh 2\alpha_o - 1$ yields:

$$SCF = \frac{2\sinh 2\alpha_o + \cosh 2\alpha_o - 1}{\cosh 2\alpha_o - 1}$$

But, $\sinh 2\alpha_o = 2\sinh\alpha_o \cosh\alpha_o$ and $\cosh 2\alpha_o - 1 = \sinh^2\alpha_o$. Finally,

$$SCF = 1 + 2\coth\alpha_o = 1 + 2\left(\frac{\cosh\alpha_o}{\sinh\alpha_o}\right) = 1 + 2\left(\frac{a}{b}\right)$$

Exercise 2.10

A large plate containing a circular hole of radius, $R := 1.0$ inch has a straight slot *on one side only,* terminating in a smooth root radius of $\rho := 0.05$ inch. (a) If the overall length of the slot, $a := 0.2$ inch, estimate the stress concentration factor at the tip of the slot. (b) If the slot is long relative to the diameter of the hole, estimate the stress concentration factor at the tip of the slot?

Solution:
a) The SCF for a hole of any radius is 3. The SCF for a notch can be estimated from Inglis as

$$SCF_{notch} := 1 + 2\sqrt{\frac{a}{\rho}}$$

Using the compounding argument of Inglis, the combined SCF is:

$$SCF_{combined} := (3)SCF_{notch} \qquad \text{and} \qquad SCF_{combined} = 15$$

b) For a long notch the SCF is dominated by local features at the notch tip and the hole is merely an extension of the notch length, i.e., $a_T := R + a$. Therefore:

$$SCF_{long} := 1 + 2\sqrt{\frac{a_T}{\rho}}$$