SOLUTIONS TO CHAPTER 2 PROBLEMS

Problem 2.1

The pulley of Fig. 2.33 is composed of five portions: three cylinders (of which two are identical) and two identical cone frustum segments. The mass moment of inertia of a cylinder defined by a height *h*, a radius *R*, and mass density ρ is given by:

$$J_{cyl} = \frac{1}{2} mR^2 = \frac{1}{2}\rho h \pi R^4 \tag{P2.1}$$

whereas the mass moment of inertia of a conical portion defined by the radii R_1 and R_2, in addition to the parameters introduced in Eq. (P2.1), is expressed as:

$$J_{cone} = \frac{1}{10}\rho h \pi \left(R_1^4 + R_1^3 R_2 + R_1^2 R_2^2 + R_1 R_2^3 + R_1^4\right) \tag{P2.2}$$

As a consequence, the total mass moment of inertia of the pulley of Fig. 2.33 is calculated adding up two identical cylindrical portions of radius R_1, another cylindrical part of radius R_2, and two identical conical segments, namely:

$$J = 2\times\left(\frac{1}{2}\rho h_1 \pi R_1^4\right) + \frac{1}{2}\rho h_3 \pi R_2^4 + 2\times\frac{1}{10}\rho h_2 \pi \left(R_1^4 + R_1^3 R_2 + R_1^2 R_2^2 + R_1 R_2^3 + R_2^4\right) \tag{P2.3}$$

and the final expression of *J* is:

$$J = \frac{\pi\rho}{10}\left[10 h_1 R_1^4 + 2h_2\left(R_1^4 + R_1^3 R_2 + R_1^2 R_2^2 + R_1 R_2^3 + R_2^4\right) + 5 h_3 R_2^4\right] \tag{P2.4}$$

For the numerical values of this problem, $J = 6.543 \times 10^{-5}$ kg-m^2.

Problem 2.2

When changing the plate width from w to $w + \Delta w$, the axial mass moment of inertia becomes:

$$J_x^{'} = \frac{m'}{12}\left[\left(w+\Delta w\right)^2 + h^2\right] = \frac{\rho l\left(w+\Delta w\right)h}{12}\left[\left(w+\Delta w\right)^2 + h^2\right] \quad \text{(P2.1)}$$

where l is the plate length (dimension perpendicular on w in Fig. 2.34). Using $\Delta w = w/2$ (required by the maximum 50% width increase), changes Eq. (P2.1) to:

$$J_x^{'} = \frac{\rho l\left(w+\Delta w\right)h}{12}\left[\left(w+\Delta w\right)^2 + h^2\right] = \frac{3}{2}\times\frac{m}{12}\left(\frac{9w^2}{4} + h^2\right) \quad \text{(P2.2)}$$

This mass moment of inertia needs to be 20 times larger than the original one, namely:

$$\frac{3}{2}\times\frac{m}{12}\left(\frac{9w^2}{4} + h^2\right) = 20\times\frac{m}{12}\left(w^2 + h^2\right) \quad \text{(P 2.3)}$$

Equation (P 2.3) reduces to:

$$133w^2 + 148h^2 = 0 \quad \text{(P 2.4)}$$

which is impossible for $w > 0$ and $h > 0$.

The second modality is to calculate the mass moment of inertia with respect to an axis parallel to the central axis x and which is at a distance d from x. The new mass moment of inertia is:

$$J_\Delta = J_x + md^2 = J_x + mw^2 \quad \text{(P 2.5)}$$

which used $d = w$ (the condition of maximum d). The connection between the mass moment of inertia of Eq. (P 2.5) and the needed moment of inertia is:

$$J_x + mw^2 = 20J_x \quad \text{(P 2.6)}$$

Equation (P 2.6) can also be written as:

$$mw^2 = 19J_x \quad \text{(P 2.7)}$$

Considering that:

$$J_x = \frac{m}{12}\left(w^2 + h^2\right) \quad \text{(P 2.8)}$$

in conjunction with Eq. (P 2.7), leads to:

$$7w^2 + 19h^2 = 0 \quad \text{(P 2.9)}$$

which is, again, impossible.

The last resort is using a combination of the two methods attempted thus far. One variant is to calculate the mass moment of inertia with respect to a moved axis and to increase the plate width at the same time. Assuming the width increases by *w*/2, let us calculate the distance *d* between the central axis *x* and the new axis. Taking into account that the new mass is 3/2*m* (with *m* being the original mass), the following equation relates the new mass moment of inertia and the necessary one:

$$\frac{3}{2}\times\frac{m}{12}\left(\frac{9w^2}{4}+h^2\right)+\frac{3}{2}md^2=20\times\frac{m}{12}\left(w^2+h^2\right) \quad \text{(P 2.10)}$$

After some algebra, Eq. (P 2.10) yields the following distance *d*:

$$d=\frac{\sqrt{133w^2+148h^2}}{12} \quad \text{(P 2.11)}$$

This value is acceptable provided *d* is less or equal to *w*, which leads to:

$$h\leq\sqrt{\frac{11}{148}}w \quad \text{(P 2.12)}$$

Problem 2.3

As shown in the springs' section of Chapter 2, the stiffness of a torsional helical spring is:

$$k_h = \frac{Ed^4}{64nR}\left(1+\frac{2G}{E}\right) \quad \text{(P2.1)}$$

whereas the stiffness of a spiral (planar) spring is:

$$k_s = \frac{E\pi d^4}{64l} \quad \text{(P2.2)}$$

E is Young's modulus, G is the shear modulus, d is the wire diameter, n is the number of active turns, and R is the helical spring external radius. The length of either of the two springs is:

$$l = n(2\pi R) \quad \text{(P2.3)}$$

Using the relationship between G and E, which is given in Appendix D and which is $G = E/[2(1 + \upsilon)]$, it follows that:

$$\frac{2G}{E} = \frac{2\times\frac{E}{2(1+\nu)}}{E} = \frac{1}{1+\nu} \quad \text{(P2.4)}$$

where μ is Poisson's ratio. The helical spring stiffness becomes:

$$k_h = \frac{Ed^4}{32l}\times\frac{2+\nu}{1+\nu} \quad \text{(P2.5)}$$

The following relative stiffness ratio is formulated by means of Eqs. (P2.2) and (P2.5) as:

$$\frac{k_h - k_s}{k_h} = \frac{3+\mu}{2(2+\mu)} \quad \text{(P2.6)}$$

Figure P2.1 shows the variation of this relative stiffness ratio with Poisson's ratio, which indicates a reduction comprised between 70% and a little more than 72% by using the spiral springs instead of the helical ones.

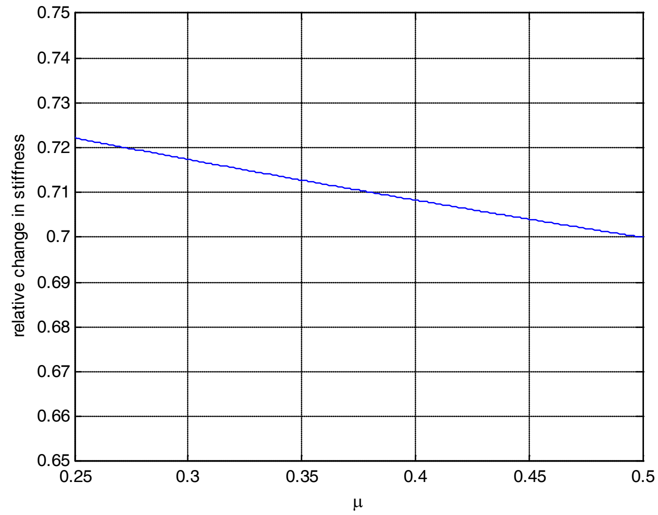

Figure P2.1 Relative stiffness change in terms of material Poisson's ratio

Problem 2.4

The springs of stiffnesses k_1 and k_2 are connected in series; therefore, their equivalent stiffness is:

$$k_{12} = \frac{k_1 k_2}{k_1 + k_2} \tag{P2.1}$$

which becomes:

$$k_{12}^{'} = \left(\frac{l}{3l}\right)^2 k_{12} = \frac{k_1 k_2}{9\left(k_1 + k_2\right)} \tag{P2.2}$$

when transferred to point A. Similarly, the original stiffness k_3 is transferred at A and becomes:

$$k_3^{'} = \left(\frac{2l}{3l}\right)^2 k_3 = \frac{4}{9} k_3 \tag{P2.3}$$

The two equivalent springs that are now located at point A are connected in parallel (they undergo identical motions), and therefore the total stiffness at point A is:

$$k_A = k_{12}^{'} + k_3^{'} = \frac{1}{9}\left(\frac{k_1 k_2}{k_1 + k_2} + 4k_3\right) \tag{P2.4}$$

Its numerical value is k_A = 32.17 N/m.

Problem 2.5

The translatory damping coefficient accounts for the lateral friction, and therefore, its expression can be retrieved from the first Eq. (2.28) taking into account that the hydraulic resistance is zero, namely:

$$c_t = \frac{2\pi\mu D_i l}{D_o - D_i} = \frac{f_d}{v} \tag{P2.1}$$

which enables expressing the coefficient of dynamic viscosity as:

$$\mu = \frac{\left(D_o - D_i\right) f_d}{2\pi D_i l v} \tag{P2.2}$$

The rotary damping coefficient is expressed in the second Eq. (2.28) as:

$$c_r = \frac{\pi\mu D_i^3 l}{2\left(D_o - D_i\right)} = \frac{m_d}{\frac{\pi n}{30}} \tag{P2.3}$$

This last equation enables formulating the coefficient of dynamic viscosity as:

$$\mu = \frac{60\left(D_o - D_i\right) m_d}{\pi^2 n D_i^3 l} \tag{P2.4}$$

By equating the coefficients of dynamic viscosity of Eqs. (P2.2) and (P2.4), the piston diameter becomes:

$$D_i = 2\sqrt{\frac{30 m_d v}{\pi n f_d}} \tag{P2.5}$$

with a numerical value of 13.3 mm, which is used in Eq. (P2.2) to calculate the coefficient of dynamic viscosity μ = 0.0298 N-s/m^2.

Problem 2.6

On each shaft, the bearings act as rotary dampers in parallel; the damping coefficients of the long and short bearings are according to the second Eq. (2.28):

$$c_l = \frac{\pi\mu D_i^3 l_1}{2\left(D_o - D_i\right)};\ c_s = \frac{\pi\mu D_i^3 l_2}{2\left(D_o - D_i\right)} \tag{P2.1}$$

Transferring the damping coefficient corresponding to the short bearings, c_s, from the original shaft to the long shaft results in an equivalent (total) damping coefficient:

$$c_e = 3c_l + 2c_s \frac{N_1^2}{N_2^2} \tag{P2.2}$$

Taking into account Eq. (2.59), the angular velocity of the long shaft ω_1 is expressed in terms of the rpm (rotations-per-minute) of the short shaft n_2 as:

$$\omega_1 = \frac{N_2}{N_1} \times \omega_2 = \frac{N_2}{N_1} \times \frac{\pi n_2}{30} \tag{P2.3}$$

The damping torque applied to the long shaft is:

$$m_d = c_e \omega_1 = \left(3c_l + 2c_s \frac{N_1^2}{N_2^2}\right) \times \frac{N_2}{N_1} \times \frac{\pi n_2}{30} \tag{P2.4}$$

With the numerical data of this example, the following results are obtained: c_l = 9.4248 x 10^{-8} N-m-s, c_s = 6.2832 x 10^{-8} N-m-s, c_e = 5.7453 x 10^{-7} N-m-s, and m_d = 7.8967 x 10^{-6} N-m.

Problem 2.7

Figure P2.1 shows the side view of the plate in an arbitrary transverse position defined by the coordinate z in the channel.

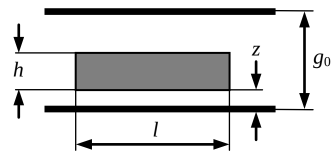

Figure P2.1 Transverse position of body sliding in a channel

Friction, which leads to viscous damping, occurs in both interstices, underneath and above the sliding body; according to Eq. (2.27), the corresponding viscous damping coefficients are:

$$c_1 = \frac{\mu l w}{z};\ c_2 = \frac{\mu l w}{g_0 - h - z} \tag{P2.1}$$

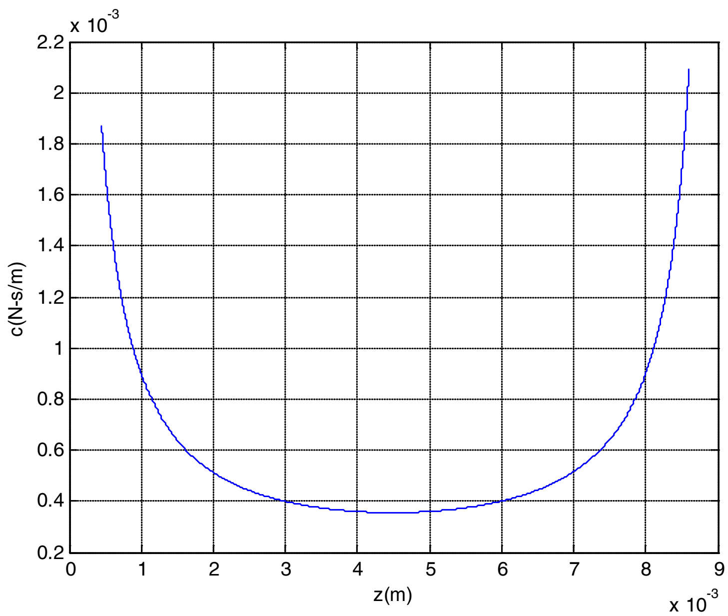

Figure P2.2 Equivalent viscous damping coefficient as a function of the body-channel wall gap

The total effect on the body is that of two dampers that are connected in parallel, and the equivalent damping coefficient, as provided in Eq. (2.31), is:

$$c = c_1 + c_2 = \frac{\mu lw\left(g_0 - h\right)}{z\left(g_0 - h - z\right)} \tag{P2.2}$$

Figure P2.2 illustrates the variation of *c* with the position *z* of the body in the channel. For the plot of Fig. P2.2, the minimum and maximum values of *z* have been set as

$$z_{\min} = \frac{g_0 - h}{20};\ z_{\max} = g_0 - h - z_{\min} \tag{P2.3}$$

whose values are: z_{min} = 0.45 mm and z_{max} = 8.6 mm.

The damping coefficient reaches its maximum value when the gap between the body and the internal wall of the channel is minimum (which means the other gap is maximum) or maximum (which means the other gap is minimum). In order to find the transverse positions for which *c* is minimum, we need to analyze the first derivative of *c* with respect to *z*, which is:

$$\frac{dc}{dz} = -\frac{\mu lw\left(g_0 - h\right)\left(g_0 - h - 2z\right)}{z^2\left(g_0 - h - z\right)^2} \tag{P2.4}$$

The derivative of Eq. (P2.4) is zero for $z' = \frac{g_0 - h}{2} = 4.5\text{ mm}$; it can be checked that *c* is minimum for that value because for *z* < *z*' the derivative of Eq. (P2.4) is negative (so the function *c* decreases) whereas for *z* > *z*' the derivative is positive (which indicates *c* increases).

Problem 2.8

Let us use Newton's second law of motion based on Fig. P2.1 below.

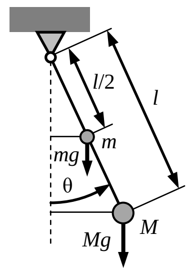

Figure P2.1 Rod-bob pendulum in arbitrary position with geometry and forces

The gravity forces of the bob and of the rod (which is placed at the center of gravity) produce moments opposing the rotation θ, according to Newton's second law of motion:

$$J\ddot{\theta} = -Mgl\sin\theta - mg\frac{l}{2}\sin\theta \qquad \text{(P2.1)}$$

with $m = \rho l \pi d^2/4$ being the total mass of the rod. For small rotations, sinθ is approximately equal to θ and, therefore, Eq. (P2.1) becomes:

$$J\ddot{\theta} = -Mgl\theta - mg\frac{l}{2}\theta \qquad \text{(P2.2)}$$

The total moment of inertia of the mechanical system about the rotation axis passing through the pivot point is:

$$J = Ml^2 + \frac{1}{3}ml^2 \qquad \text{(P2.3)}$$

Substitution of Eq. (P2.3) into Eq. (P2.2) results, after simplification, in:

$$l\left(M + \frac{m}{3}\right)\ddot{\theta} + \left(M + \frac{m}{2}\right)g\theta = 0 \qquad \text{(P2.4)}$$

It is known that Eq. (P2.4) can be written in the generic form:

$$\ddot{\theta} + \omega_n^2\theta = 0 \qquad \text{(P2.5)}$$

and therefore the natural frequency corresponding to Eq. (P2.4) is:

$$\omega_n = \sqrt{\frac{\left(M + \frac{m}{2}\right)g}{\left(M + \frac{m}{3}\right)l}} = \sqrt{\frac{3(2M + m)g}{2(3M + m)l}} = \sqrt{\frac{3\left(8M + \pi\rho l d^2\right)g}{2\left(12M + \pi\rho l d^2\right)l}} \tag{P2.6}$$

When the mass of the rod is not considered (which is equivalent to $\rho = 0$), Eq. (P2.6) reduces to $\omega_n^* = \sqrt{\frac{g}{l}}$; this is the known equation of the natural frequency of a massless rod – bob pendulum. For the numerical values of this problem, the natural frequencies of interest are $\omega_n = 18.097 \text{ rad/s}$ and $\omega_n^* = 18.083 \text{ rad/s}$ so the relative error between these two values is less than 0.08%.

Problem 2.9

Consider a counterclockwise rotation of the lower lever by an angle θ, as sketched in Fig. P2.1, which shows the a displaced position of the two-lever system.

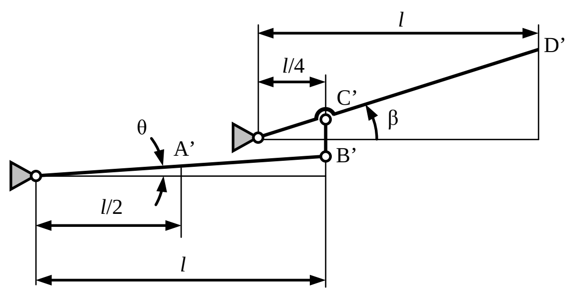

Figure P2.1 Two-lever mechanical system in displaced position

The spring is transferred from its original location (A in Fig. 2.39) at the end point B, case where its stiffness becomes:

$$k' = \left(\frac{l/2}{l}\right)^2 k = \frac{k}{4} \qquad \text{(P2.1)}$$

Similarly, the mass is relocated from its original position at D to C (which is the same as being moved at B on the adjacent rod) and the transformed mass becomes:

$$m' = \left(\frac{l}{l/4}\right)^2 m = 16m \qquad \text{(P2.2)}$$

For small motions, the dynamic equation describing the rotation of the left rod about its pivot point is:

$$J\ddot{\theta} = -f_e l \text{ or } m' l^2 \ddot{\theta} = -k'(l\theta)l \qquad \text{(P2.3)}$$

where f_e is the spring elastic force. Combining Eqs. (P2.1), (P2.2) and (P2.3) yields:

$$16ml^2\ddot{\theta} + \frac{1}{4}kl^2\theta = 0 \text{ or } \ddot{\theta} + \frac{k}{64m}\theta = 0 \qquad \text{(P2.4)}$$

which indicates the natural frequency is:

$$\omega_n = \frac{1}{8}\sqrt{\frac{k}{m}} \qquad \text{(P2.5)}$$

The mass m needs to be moved to the left on the upper lever in order to increase the natural frequency. Assuming the mass moved a distance x measured from the right end of the upper lever, it can be shown that the modified dynamic equation is:

$$16m\left(l-x\right)^2\ddot{\theta}+\frac{1}{4}kl^2\theta=0 \tag{P2.6}$$

As a consequence, the new natural frequency is:

$$\omega_n^*=\frac{l}{8\left(l-x\right)}\sqrt{\frac{k}{m}} \tag{P2.7}$$

The condition of the problem actually requires that:

$$\omega_n^*=1.2\omega_n \tag{P2.8}$$

By combining Eqs. (P2.5), (P2.7) and (P2.8), results in x = l/6.

Problem 2.10

The following coordinate connections can be formulated considering small motions occur in Fig. P2.1:

$$\begin{cases} y = 2l\varphi \\ x = l\varphi = R\theta \end{cases} \tag{P2.1}$$

As a consequence, the coordinates *x*, *y* and θ can be expressed in terms of φ (the rotation angle of the horizontal rod) as:

$$\begin{cases} \theta = \dfrac{l}{R}\varphi \\ x = l\varphi \\ y = 2l\varphi \end{cases} \tag{P2.2}$$

The original mechanical system is partitioned in two subsystems as shown below in Fig. P2.1 with *f* being the force connecting the two separated portions.

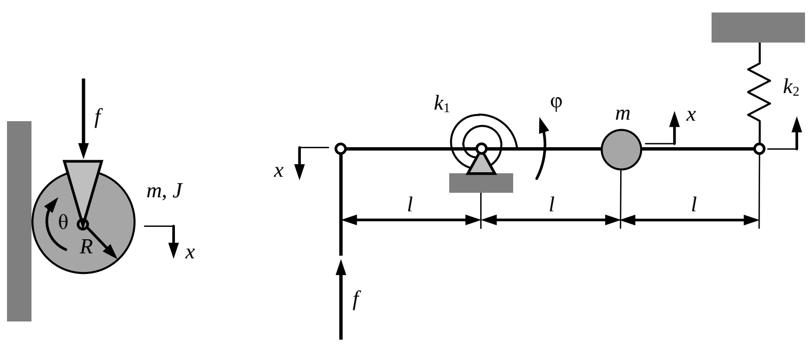

Figure P2.1 Free-body diagrams and geometric parameters

The no-slippage rolling of the wheel about the instant center of rotation (the contact point with the vertical wall) is governed by Newton's second law of rotation motion and results in the equation:

$$\left(J + mR^2\right)\ddot{\theta} = fR \tag{P2.3}$$

Taking into account that the wheel moment of inertia about its center is $J = mR^2/2$, the force *f* is expressed from Eq. (P2.3) as

$$f = \frac{3mR}{2}\ddot{\theta} \tag{P2.4}$$

Rotation of the horizontal massless rod about its pivot point is similarly expressed as