

## Chapter 1

# Overview of Finite Element Method

1.1

$$S^{(l)} = n r \quad \text{with } r = 2 R \sin\left(\frac{1}{2} \cdot \frac{2\pi}{n}\right) = 2 R \sin \frac{\pi}{n}$$

$$S^{(u)} = n s \quad \text{with } s = 2 R \tan\left(\frac{1}{2} \cdot \frac{2\pi}{n}\right) = 2 R \tan \frac{\pi}{n}$$

Using series expansions of  $\sin \theta$  and  $\tan \theta$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \dots$$

$$\tan \theta = \theta + \frac{\theta^3}{3} + \frac{2}{15} \theta^5 + \dots$$

we find

$$S^{(l)} = 2nR \left( \frac{\pi}{n} - \frac{1}{6} \frac{\pi^3}{n^3} + \dots \right)$$

$$= 2\pi R - \frac{\pi^3 R}{3n^2} + \dots$$

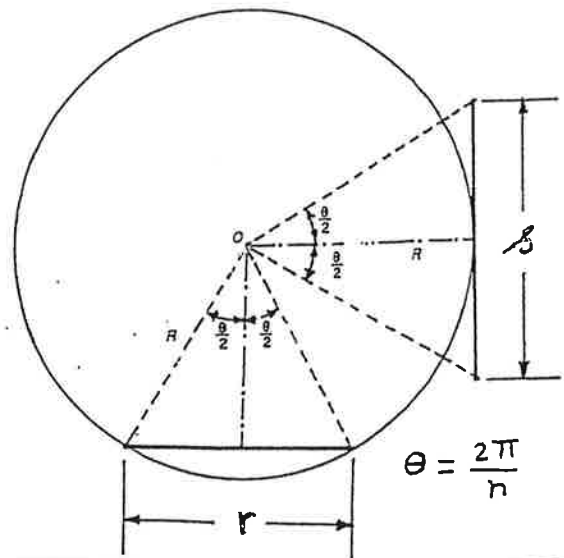
and

$$S^{(u)} = 2nR \left( \frac{\pi}{n} + \frac{1}{3} \frac{\pi^3}{n^3} + \dots \right)$$

$$= 2\pi R + \frac{2}{3} \frac{\pi^3 R}{n^2} + \dots$$

Since the true value of  $S$  is given by  $2\pi R$ , we have

$$S^{(l)} \leq S \leq S^{(u)}$$



1.2

For  $R = 1$ :

$n$	$S^{(k)} = 2nR \sin \frac{\pi}{n}$	$S^{(u)} = 2nR \tan \frac{\pi}{n}$
3	5.1962	10.3923
4	5.6569	8.0000
5	5.8779	7.2654
6	6.0000	6.9282
7	6.0744	6.7420
8	6.1229	6.6274
9	6.1564	6.5515
10	6.1803	6.4984
11	6.1981	6.4598
12	6.2117	6.4308

1.3

From Fig. 1.4 (shown below), the area of the circle corresponding to the inscribed polygon is given by

$$\begin{aligned} A^{(l)} &= n \left( \frac{1}{2} AB \times OP \right) = \frac{n}{2} \left( 2R \sin \frac{\pi}{n} \right) \left( R \cos \frac{\pi}{n} \right) \\ &= \frac{nR^2}{2} \left( 2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} \right) = \frac{1}{2} nR^2 \sin \frac{2\pi}{n} \\ &= \frac{1}{2} nR^2 \frac{2\pi}{n} \left( \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} \right) = \pi R^2 \left( \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} \right) \end{aligned}$$

As  $n \rightarrow \infty$ ,  $\frac{2\pi}{n} \rightarrow 0$  and  $A^{(l)} \Rightarrow \pi R^2 = A_{\text{correct}}$

Similarly, the area of the circle corresponding to the circumscribed polygon can be expressed as

$$\begin{aligned} A^{(u)} &= n \left( \frac{1}{2} CD \times OQ \right) = \frac{n}{2} \left( 2R \tan \frac{\pi}{n} \right) R \\ &= nR^2 \tan \frac{\pi}{n} = nR^2 \frac{\pi}{n} \left( \frac{\tan \frac{\pi}{n}}{\frac{\pi}{n}} \right) = \pi R^2 \left( \frac{\tan \frac{\pi}{n}}{\frac{\pi}{n}} \right) \end{aligned}$$

As  $n \rightarrow \infty$ ,  $\frac{\pi}{n} \rightarrow 0$  and  $A^{(u)} \Rightarrow \pi R^2 = A_{\text{correct}}$

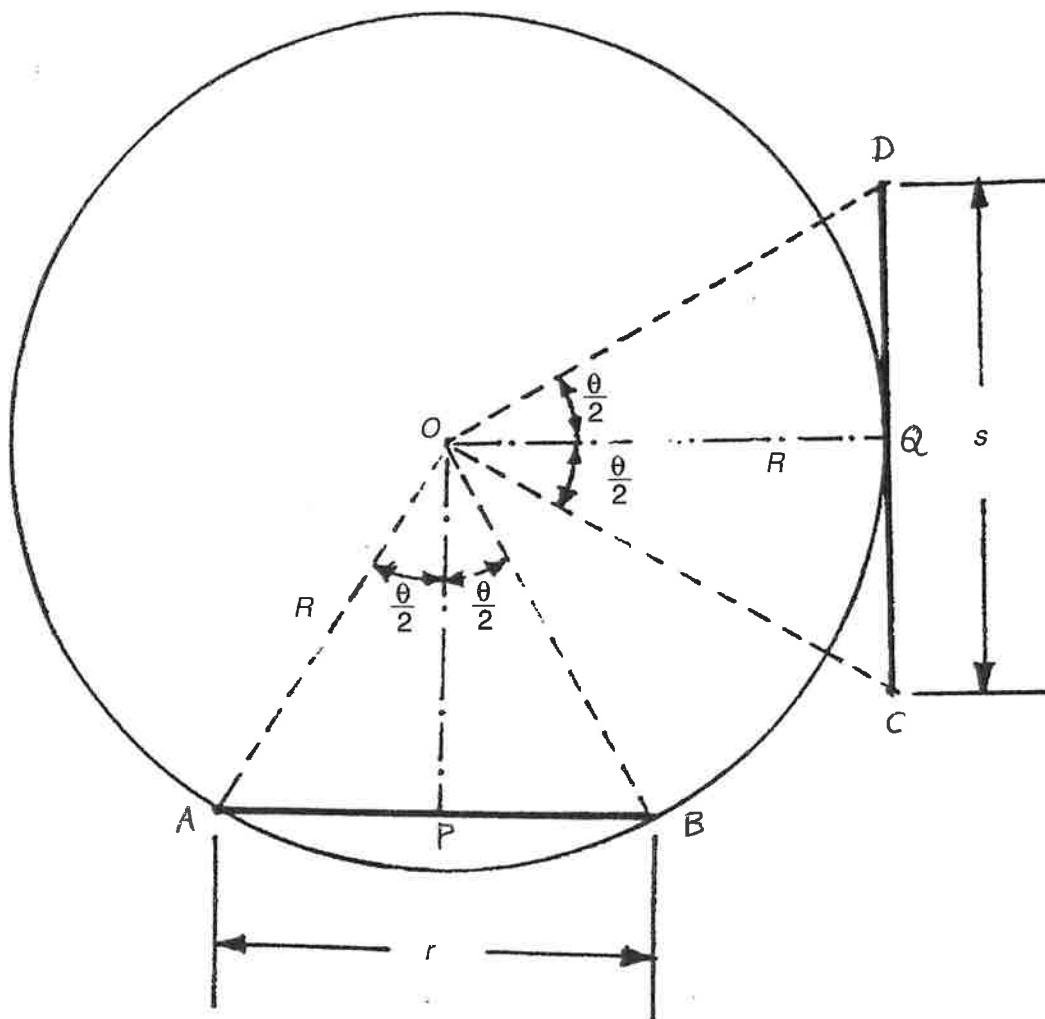
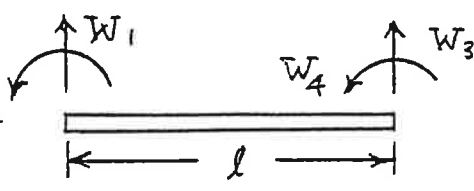


Figure 1.4.

1.4  $w(x) = W_1^{(e)} \cdot \frac{1}{l^3} (2x^3 - 3lx^2 + l^3) + W_2^{(e)} \cdot \frac{1}{l^2} (x^3 - 2lx^2 + l^2x) + W_3^{(e)} \cdot \frac{1}{l^3} (3lx^2 - 2x^3) + W_4^{(e)} \cdot \frac{1}{l^2} (x^3 - lx^2)$



$$[K^{(e)}] = \frac{2EI}{l^3} \begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix} = [K_{\sim}]$$

$$\vec{W}^{(e)} = \begin{Bmatrix} W_1^{(e)} \\ W_2^{(e)} \\ W_3^{(e)} \\ W_4^{(e)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ W_3 \\ W_4 \end{Bmatrix} = \vec{W}_{\sim}$$

$$\vec{P}^{(e)} = \begin{Bmatrix} P_1^{(e)} \\ P_2^{(e)} \\ P_3^{(e)} \\ P_4^{(e)} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P \\ 0 \end{Bmatrix} = \vec{P}_{\sim}$$

$[K_{\sim}] \vec{W}_{\sim} = \vec{P}_{\sim}$  gives

$$\frac{2EI}{l^3} \begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P \\ 0 \end{Bmatrix} \quad (1)$$

The third and fourth equations in (1) are given by

$$\frac{2EI}{l^3} (6W_3 - 3lW_4) = P \quad (2)$$

$$\frac{2EI}{l^3} (-3lW_3 + 2l^2W_4) = 0 \quad (3)$$

Solution of Eqs. (2) and (3) is:

$$W_3 = \frac{Pl^3}{3EI}, \quad W_4 = \frac{Pl^2}{2EI}$$

The deflection within the element can be expressed as

$$w(x) = \frac{P}{3EI} (3lx^2 - 2x^3) + \frac{P}{2EI} (x^3 - lx^2)$$

$$\frac{d^2w}{dx^2}(x) = \frac{P}{3EI} (6l - 12x) + \frac{P}{2EI} (6x - 2l) = \frac{P}{EI} (l - x)$$

$$\text{Bending moment} = M(x) = EI \frac{d^2w}{dx^2} = P(l - x)$$

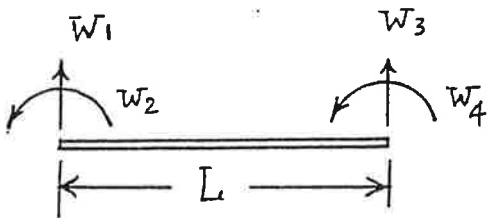
$$\text{stress in the element} = \sigma(x) = \frac{M(x) \cdot y}{I}$$

where  $y$  = distance from neutral axis.

$$\therefore \sigma(x) = \frac{P}{I} (l - x) \cdot y \quad ; \quad l \equiv L$$

1.5

$$[K^{(e)}] = [K] = \frac{2EI}{L^3} \begin{bmatrix} 6 & -3L & -6 & 3L \\ 3L & 2L^2 & 3L & -L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{matrix}$$



$$[K] \vec{W} = \vec{P} \quad \text{where} \quad [K] = \frac{2EI}{L^3} \begin{bmatrix} 6 & -3L \\ -3L & 2L^2 \end{bmatrix}$$

$$\text{with} \quad \vec{W} = \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} \quad \text{and} \quad \vec{P} = \begin{Bmatrix} 0 \\ M_0 \end{Bmatrix}$$

These equations can be rewritten as

$$6W_3 - 3LW_4 = 0, \quad -3LW_3 + 2L^2W_4 = \frac{M_0 L^3}{2EI}$$

Solution:  $W_3 = \frac{M_0 L^2}{2EI}, \quad W_4 = \frac{M_0 L}{EI}$

Bending moment in element =  $M(x) = EI \frac{d^2 w}{dx^2}$

with  $w(x) = W_3 \cdot \frac{1}{L^3} (3Lx^2 - 2x^3) + W_4 \cdot \frac{1}{L^2} (x^3 - Lx^2)$

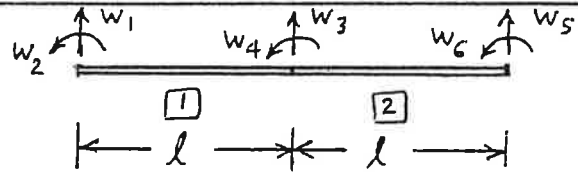
and  $\frac{d^2 w}{dx^2}(x) = W_3 \cdot \frac{1}{L^3} (6L - 12x) + \frac{W_4}{L^2} (6x - 2L)$

$$= \frac{M_0 L^2}{2EI} \cdot \frac{1}{L^3} (6L - 12x) + \frac{M_0 L}{EI} \cdot \frac{1}{L^2} (6x - 2L)$$

$$= \frac{M_0}{EI}$$

$$\sigma(x) = \text{stress} = \frac{M(x) \cdot y}{I} = \left( EI \frac{d^2 w}{dx^2} \right) \frac{y}{I} = \frac{M_0 \cdot y}{I}$$

1.6 Assembling the element stiffness matrices by retaining only the non-zero degrees of freedom, we obtain



$$w_1 = w_2 = w_5 = 0$$

$$[K] = \frac{2EI}{L^3} \begin{bmatrix} 6+6 & -3l+3l & 3l \\ -3l+3l & 2l^2+2l^2 & l^2 \\ 3l & l^2 & 2l^2 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \\ w_6 \end{matrix} ; \quad l = \frac{L}{2}$$

$$= \frac{16EI}{L^3} \begin{bmatrix} 12 & 0 & \frac{3}{2}L \\ 0 & L^2 & \frac{1}{4}L^2 \\ \frac{3}{2}L & \frac{1}{4}L^2 & \frac{1}{2}L^2 \end{bmatrix}$$

Equilibrium equations:

$$\frac{4EI}{L^3} \begin{bmatrix} 48 & 0 & 6L \\ 0 & 4L^2 & L^2 \\ 6L & L^2 & 2L^2 \end{bmatrix} \begin{Bmatrix} w_3 \\ w_4 \\ w_6 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \end{Bmatrix} \quad (1)$$

Scalar form of Eq. (1):

$$48 W_3 + 6L W_6 = \frac{PL^3}{4EI} \quad (2)$$

$$4L^2 W_4 + L^2 W_6 = 0 \Rightarrow W_4 = -\frac{W_6}{4} \quad (3)$$

$$6L W_3 + L^2 W_4 + 2L^2 W_6 = 0 \quad (4)$$

Eqs. (3) and (4) give

$$W_3 = -\frac{L^2}{6L} W_4 - \frac{2L^2}{6L} W_6 = -\frac{7}{24} L W_6 \quad (5)$$

Eqs. (2), (5) and (3) yield

$$W_3 = \frac{7}{768} \frac{PL^3}{EI}, \quad W_4 = \frac{1}{128} \frac{PL^2}{EI}, \quad W_6 = -\frac{PL^2}{32EI} \quad (6)$$

$$\text{Bending moment in element} = M(x) = EI \frac{d^2 w}{dx^2}$$

$$\text{stress in element} = \sigma(x) = \frac{M(x) \cdot y}{I} = E y \frac{d^2 w}{dx^2} \quad (7)$$

$$\sigma_{xx}^{(1)} = -E y \left[ \frac{1}{L^3} (6L - 12x) \frac{7PL^3}{768EI} + \frac{1}{L^2} (6x - 2L) \frac{PL^2}{128EI} \right]$$

$$= -y \frac{P}{I} \left( \frac{5L - 8x}{128} \right) \quad (7)$$

$$\sigma_{xx}^{(2)} = -y E \left[ \frac{1}{L^3} (12x - 6L) \frac{7PL^3}{768EI} + \frac{1}{L^2} (6x - 4L) \frac{PL^2}{128EI} \right]$$

$$+ \frac{1}{L^2} (6x - 2L) \left( -\frac{PL^2}{32EI} \right) \Big]$$

$$= y \frac{P}{I} \left( \frac{x + 2L}{32} \right) \quad (8)$$

1.7

Assembled stiffness matrix,  $[K]$ , is same as the one derived in Problem 1.6. Equilibrium equations:

$$\frac{4EI}{L^3} \begin{bmatrix} 48 & 0 & 6L \\ 0 & 4L^2 & L^2 \\ 6L & L^2 & 2L^2 \end{bmatrix} \begin{Bmatrix} W_3 \\ W_4 \\ W_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_0 \\ 0 \end{Bmatrix} \quad (1)$$

Solution of Eqs. (1):

$$W_3 = \frac{M_0 L^2}{128 EI}, \quad W_4 = \frac{5 M_0 L}{64 EI}, \quad W_6 = -\frac{M_0 L}{16 EI}$$

Stress element 1:

$$\begin{aligned} \sigma(x) &= -E y \frac{d^2 w}{dx^2} = -E y \left[ \frac{W_3}{L^3} (6L - 12x) + \frac{W_4}{L^2} (6x - 2L) \right] \\ &= -E y \left[ \frac{M_0 L^2}{128 EI} \cdot \frac{8}{L^3} (3L - 12x) + \frac{5 M_0 L}{64 EI} \cdot \frac{4}{L^2} (6x - L) \right] \end{aligned}$$

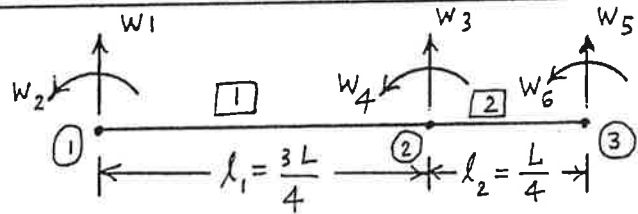
$$= -\frac{M_0 y}{8 I L} (9x - L)$$

Stress in element 2:

$$\begin{aligned} \sigma(x) &= -E y \left[ \frac{W_3}{l^3} (12x - 6l) + \frac{W_4}{l^2} (6x - 4l) + \frac{W_6}{l^2} (6x - 2l) \right] \\ &= -E y \left[ \frac{M_0 L^2}{128 E I} \cdot \frac{8}{L^3} (12x - 3L) + \frac{5 M_0 L}{64 E I} \cdot \frac{4}{L^2} (6x - 2L) \right. \\ &\quad \left. - \frac{M_0 L}{16 E I} \cdot \frac{4}{L^2} (6x - L) \right] \\ &= -\frac{9 M_0 y}{16 I L} (2x - L) \end{aligned}$$

1.8

2 element idealization  $\Rightarrow$



$$[K^{(e)}] = \frac{2EI}{l_e^3} \begin{bmatrix} 6 & 3l_e & -6 & 3l_e \\ 3l_e & 2l_e^2 & -3l_e & l_e^2 \\ -6 & -3l_e & 6 & -3l_e \\ 3l_e & l_e^2 & -3l_e & 2l_e^2 \end{bmatrix}$$

where  $l_e =$  length of element  $e$ .

Boundary conditions:  $W_1 = W_2 = W_5 = W_6 = 0$ .

Considering only the free degrees of freedom, the assembled stiffness matrix of the system can be obtained as

$$[K] = 2EI \begin{bmatrix} \frac{6}{l_1^3} + \frac{6}{l_2^3} & -\frac{3}{l_1^2} + \frac{3}{l_2^2} \\ -\frac{3}{l_1^2} + \frac{3}{l_2^2} & \frac{2}{l_1} + \frac{2}{l_2} \end{bmatrix} \begin{matrix} W_3 \\ W_4 \end{matrix} = \begin{bmatrix} \left(\frac{7168}{9} \frac{EI}{L^3}\right) & \left(\frac{256}{3} \frac{EI}{L^2}\right) \\ \left(\frac{256}{3} \frac{EI}{L^2}\right) & \left(\frac{64}{3} \frac{EI}{L}\right) \end{bmatrix}$$

Equilibrium equations:  $[K] \vec{W} = \vec{P}$

ie.

$$\begin{bmatrix} \frac{7168}{9} \frac{EI}{L^3} & \frac{256}{3} \frac{EI}{L^2} \\ \frac{256}{3} \frac{EI}{L^2} & \frac{64}{3} \frac{EI}{L} \end{bmatrix} \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Solution is:

$$W_3 = \frac{9}{4096} \frac{PL^3}{EI}, \quad W_4 = -\frac{9}{1024} \frac{PL^2}{EI}$$

Stress in element 1:

$$w^{(1)}(x) = \frac{9}{4096} \frac{PL^3}{EI} \cdot \frac{1}{l_1^3} (3l_1 x^2 - 2x^3) - \frac{9}{1024} \frac{PL^2}{EI} \cdot$$

$$\frac{1}{l_1^2} (x^3 - l_1 x^2)$$

$$= \frac{P}{192 EI} \left( \frac{9}{2} L x^2 - 5 x^3 \right)$$

$$\sigma^{(1)}(x) = \frac{M^{(1)}(x)}{I} = E y \frac{d^2 w^{(1)}}{dx^2} = E y \cdot \frac{P}{192 EI} (9L - 30x)$$

$$= \frac{P}{64 I} (3L - 10x) y.$$

Stress in element 2:

$$w^{(2)}(x) = \frac{9}{4096} \frac{PL^3}{EI} \left( \frac{1}{l_2^3} \right) (2x^3 - 3l_2 x^2 + l_2^3)$$

$$- \frac{9}{1024} \frac{PL^2}{EI} \left( \frac{1}{l_2^2} \right) (x^3 - 2l_2 x^2 + l_2^2 x)$$

$$= \frac{9}{64} \frac{P}{EI} \left( x^3 - \frac{L}{4} x^2 - \frac{L^2}{16} x + \frac{L^3}{64} \right)$$

$$\sigma^{(2)}(x) = \frac{M^{(2)}(x)}{I} = E y \frac{d^2 w^{(2)}}{dx^2} = E y \cdot \frac{9P}{128 EI} (12x - L)$$

$$= \frac{9P}{128 I} (12x - L) y.$$

1.9

$$A(x) = A_0 e^{-x/l}$$

$$\pi = \text{strain energy} = \frac{1}{2} \int_0^l \sigma \epsilon A(x) \cdot dx$$

$$= \frac{1}{2} \int_0^l E \left( \frac{\partial u}{\partial x} \right)^2 A(x) dx \quad \text{with } u(x) = \left(1 - \frac{x}{l}\right) U_1 + \left(\frac{x}{l}\right) U_2$$

$$\pi = \frac{1}{2} \int_0^l E \left( \frac{U_2 - U_1}{l} \right)^2 A_0 e^{-x/l} dx$$

$$= \frac{1}{2} E A_0 \left( \frac{U_2 - U_1}{l} \right)^2 \int_0^l e^{-x/l} dx$$

$$= \frac{1}{2} E A_0 \left( \frac{U_2 - U_1}{l} \right)^2 (0.6321 l) = \frac{1}{2} \{U_1 \quad U_2\} [K] \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}$$

with

$$[K] = \frac{0.6321 E A_0}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$T = \text{kinetic energy} = \frac{1}{2} \int_0^l \rho \left( \frac{\partial u}{\partial t} \right)^2 A(x) \cdot dx$$

$$= \frac{1}{2} \int_0^l \{ \dot{U}_1 \quad \dot{U}_2 \} \begin{bmatrix} \left(1 - \frac{x}{l}\right)^2 & \left(1 - \frac{x}{l}\right) \frac{x}{l} \\ \left(1 - \frac{x}{l}\right) \frac{x}{l} & \left(\frac{x}{l}\right)^2 \end{bmatrix} \begin{Bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{Bmatrix} A_0 e^{-x/l} dx$$

$$= \frac{1}{2} \dot{U}^T [M] \dot{U}$$

$$\text{Using } \int_0^l e^{-x/l} \cdot dx = 0.6321 l,$$

$$\int_0^l \frac{x}{l} e^{-x/l} \cdot dx = 0.2642 l,$$

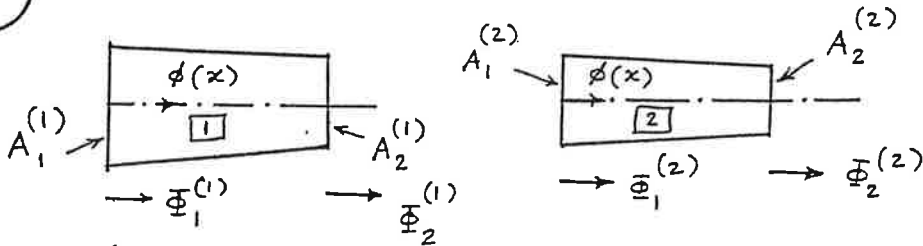
$$\text{and } \int_0^l \frac{x^2}{l^2} e^{-x/l} \cdot dx = 0.1605 l, \text{ we obtain}$$

$$[M] = \rho A_0 l \begin{bmatrix} 0.2642 & 0.1037 \\ 0.1037 & 0.1605 \end{bmatrix}$$

where  $\rho$  = density and  $l$  = length of element.

1.10

Step 1. Idealization:



$$l^{(i)} = 5 \text{ cm}, E^{(i)} = 2 \times 10^7 \text{ N/cm}^2 ; i = 1, 2$$

$$A_1^{(1)} = 2 \text{ cm}^2, A_2^{(1)} = A_1^{(2)} = 1.5 \text{ cm}^2, A_2^{(2)} = 1 \text{ cm}^2$$

Step 2. Interpolation model:

$$\phi(x) = \left[ \left(1 - \frac{x}{l^{(e)}}\right) \quad \left(\frac{x}{l^{(e)}}\right) \right] \begin{Bmatrix} \Phi_1^{(e)} \\ \Phi_2^{(e)} \end{Bmatrix}$$

Step 3. Element equations:

$$\text{strain energy of element } e = \int_0^{l^{(e)}} \left( \frac{1}{2} \sigma^{(e)} \epsilon^{(e)} \right) A^{(e)} dx$$

$$\text{with } \sigma^{(e)} = E^{(e)} \epsilon^{(e)}, \quad \epsilon^{(e)} = \frac{d\phi}{dx} = \frac{\Phi_2^{(e)} - \Phi_1^{(e)}}{l^{(e)}}$$

$$\text{Here } A^{(e)}(x) = A_1^{(e)} + \frac{A_2^{(e)} - A_1^{(e)}}{l^{(e)}} x$$

$$\text{Hence } \pi^{(e)} = \text{strain energy} = \frac{E^{(e)}}{2} \left( \frac{\Phi_2^{(e)} - \Phi_1^{(e)}}{l^{(e)}} \right)^2 \int_0^{l^{(e)}} \left\{ A_1^{(e)} + \frac{A_2^{(e)} - A_1^{(e)}}{l^{(e)}} x \right\} dx$$

$$= \frac{A_{av}^{(e)} E^{(e)}}{2 l^{(e)}} \left( \Phi_1^{(e)2} + \Phi_2^{(e)2} - 2 \Phi_1^{(e)} \Phi_2^{(e)} \right)$$

$$\text{where } A_{av}^{(e)} = \text{average area} = \frac{A_1^{(e)} + A_2^{(e)}}{2}$$

$$\text{Expressing } \pi^{(e)} \text{ as } \pi^{(e)} = \frac{1}{2} \vec{\Phi}^{(e)T} [K^{(e)}] \vec{\Phi}^{(e)},$$

we obtain

$$[K^{(e)}] = \frac{A_{av}^{(e)} E^{(e)}}{l^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{with } \vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1^{(e)} \\ \Phi_2^{(e)} \end{Bmatrix}$$

$$A_{av}^{(1)} = \frac{2 + 1.5}{2} = 1.75 \text{ cm}^2, \quad A_{av}^{(2)} = \frac{1.5 + 1}{2} = 1.25 \text{ cm}^2$$

$$[K^{(1)}] = 10^6 \begin{bmatrix} 7 & -7 \\ -7 & 7 \end{bmatrix}, \quad [K^{(2)}] = 10^6 \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$$

Step 4. Assembly of equations:

$$[K_{\sim}] = 10^6 \begin{bmatrix} 7 & -7 & 0 \\ -7 & 7+5 & -5 \\ 0 & -5 & 5 \end{bmatrix}$$

$$\vec{\Phi}_{\sim} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix}, \quad \vec{P}_{\sim} = \begin{Bmatrix} P_1 \\ 0 \\ 1 \end{Bmatrix}, \quad P_1 = \text{reaction at node 1}$$

$$[K_{\sim}] \vec{\Phi}_{\sim} = \vec{P}_{\sim} \quad (1)$$

Step 5. Apply boundary conditions and solve equations:

since  $\Phi_1 = 0$ , we delete first row and first column in Eqs. (1) to obtain

$$[K] \vec{\Phi} = \vec{P} \Rightarrow 10^6 \begin{bmatrix} 12 & -5 \\ -5 & 5 \end{bmatrix} \begin{Bmatrix} \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad (2)$$

Solution of Eqs. (2) gives

$$\Phi_2 = 1.428 \times 10^{-7} \text{ cm}, \quad \Phi_3 = 3.428 \times 10^{-7} \text{ cm}$$

Step 6. Find stresses:

$$\begin{aligned} \sigma^{(1)} &= E^{(1)} \epsilon^{(1)} = E^{(1)} \frac{\Phi_2^{(1)} - \Phi_1^{(1)}}{l^{(1)}} = (2 \times 10^7) \left( \frac{1.428 \times 10^{-7} - 0}{5} \right) \\ &= 0.5712 \text{ N/cm}^2 \end{aligned}$$

$$\begin{aligned} \sigma^{(2)} &= E^{(2)} \epsilon^{(2)} = E^{(2)} \frac{\Phi_2^{(2)} - \Phi_1^{(2)}}{l^{(2)}} \\ &= (2 \times 10^7) \left( \frac{3.428 \times 10^{-7} - 1.428 \times 10^{-7}}{5} \right) = 0.8 \text{ N/cm}^2 \end{aligned}$$

1.11 (a) Element characteristic matrices:

$$c_1 = \frac{\pi d_1^4}{128 \mu l_1} = \frac{\pi (5^4)}{128 (1.6 \times 10^{-6})(1000)} = 9.5874 \times 10^3 \text{ in}^5/\text{lb-sec}$$

$$c_2 = \frac{\pi d_2^4}{128 \mu l_2} = \frac{\pi (2^4)}{128 (1.6 \times 10^{-6})(1500)} = 0.1636 \times 10^3 \text{ in}^5/\text{lb-sec}$$

$$c_3 = \frac{\pi d_3^4}{128 \mu l_3} = \frac{\pi (4^4)}{128 (1.6 \times 10^{-6})(2000)} = 1.9635 \times 10^3 \text{ in}^5/\text{lb-sec}$$

Pipe element 1 with nodes 1 and 2:

$$[K^{(1)}] = c_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 9587.4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix}$$

Pipe element 2 with nodes 2 and 3:

$$[K^{(2)}] = c_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1636.0 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 2 & 3 \\ 3 & 2 \end{matrix}$$

Pipe element 3 with nodes 2 and 4:

$$[K^{(3)}] = c_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1963.5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 2 & 4 \\ 4 & 2 \end{matrix}$$

Assembled characteristic matrix:

$$[K] = \begin{bmatrix} 9587.4 & -9587.4 & 0 & 0 \\ -9587.4 & (9587.4 & -163.6 & -1963.5 \\ & +163.6 & & \\ & +1963.5) & & \\ 0 & -163.6 & 163.6 & 0 \\ 0 & -1963.5 & 0 & 1963.5 \end{bmatrix}$$

System equations:

$$\begin{bmatrix} 9587.4 & -9587.4 & 0 & 0 \\ -9587.4 & 11714.5 & -163.6 & -1963.5 \\ 0 & -163.6 & 163.6 & 0 \\ 0 & -1963.5 & 0 & 1963.5 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} \quad (1)$$

where  $Q_i$  is the external flow rate at node  $i$ .

Since the pressures at nodes 1, 3 and 4 are known,  $p_2$  is the unknown pressure in Eq. (1) with  $Q_2 = 0$ .

By using the known values,  $p_1 = 20$ ,  $p_3 = 15$  and  $p_4 = 15$  psi, Eq. (1) can be rewritten in the form

$$\begin{bmatrix} 9587.4 & 0 & 0 & 0 \\ 0 & 11714.5 & 0 & 0 \\ 0 & 0 & 163.6 & 0 \\ 0 & 0 & 0 & 1963.5 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 9587.4 * 20 \leftarrow p_1 \\ 9587.4 p_1 + 163.6 p_3 \\ + 1963.5 p_4 \\ = 22365.5 \\ 163.6 * 15 \leftarrow p_3 \\ 1963.5 * 15 \leftarrow p_4 \end{Bmatrix} \quad (2)$$

Eq. (2) gives the nodal pressures as:

$$p_1 = 20 \text{ psi}, \quad p_2 = \frac{22365.5}{11714.5} = 19.0921 \text{ psi},$$

$$p_3 = 15 \text{ psi}, \quad p_4 = 15 \text{ psi}$$

(b) Volume flow rates in pipe elements:

$$\begin{aligned} Q_1 &= c_1 (p_1 - p_2) = 9587.4 (20 - 19.0921) \\ &= 8704.323 \text{ in}^3/\text{sec} \end{aligned}$$

$$\begin{aligned} Q_2 &= c_2 (p_1 - p_2) = 1636.0 (19.0921 - 15) \\ &= 669.4676 \text{ in}^3/\text{sec} \end{aligned}$$

$$Q_3 = c_3 (p_2 - p_4) = 1963.5 (19.0921 - 15) \\ = 8034.838 \text{ in}^3/\text{sec}$$

It can be verified that the inflow rate ( $Q_1$ ) is equal to the total outflow rate ( $Q_2 + Q_3$ ).

(c) Reynolds number for flow in each of the pipe elements :

$$Re = \frac{\rho v d}{\mu} = \frac{\rho Q d}{\left(\frac{\pi d^2}{4}\right)\mu} = \frac{4 \rho Q}{\pi d \mu}$$

For pipe element 1:

$$Re = 4 \left(\frac{1}{12}\right)^4 \frac{\rho Q}{\pi \mu d} = 4 \left(\frac{1}{12}\right)^4 \frac{(1.9)(8704.3)}{\pi (1.6 \times 10^{-6})(5)} \\ = 0.126936 \times 10^6$$

For pipe element 2:

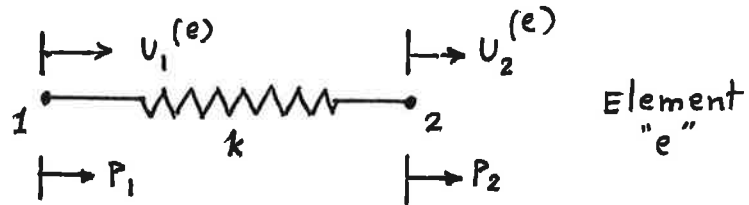
$$Re = 4 \left(\frac{1}{12}\right)^4 \frac{(1.9)(669.5)}{\pi (1.6 \times 10^{-6})(2)} = 0.024411 \times 10^6$$

For pipe element 3:

$$Re = 4 \left(\frac{1}{12}\right)^4 \frac{(1.9)(8034.8)}{\pi (1.6 \times 10^{-6})(4)} = 0.146465 \times 10^6$$

(d) Since  $Re > 2000$  in each pipe elements (segments), the flow is turbulent in all pipe elements.

1.12



When nodal forces  $P_1$  and  $P_2$  act along the nodal displacements  $U_1^{(e)}$  and  $U_2^{(e)}$ , respectively, the strain energy of the spring is given by

$$\begin{aligned}\pi &= \frac{1}{2} (\text{net force}) (\text{net displacement}) \\ &= \frac{1}{2} \{k (U_2^{(e)} - U_1^{(e)})\} (U_2^{(e)} - U_1^{(e)}) \\ &= \frac{1}{2} k (U_2^{(e)} - U_1^{(e)})^2\end{aligned}\quad (1)$$

Eq. (1) can be expressed in matrix form as

$$\pi = \frac{1}{2} \{U_1^{(e)} \quad U_2^{(e)}\} \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} U_1^{(e)} \\ U_2^{(e)} \end{Bmatrix}\quad (2)$$

or

$$\pi = \frac{1}{2} \vec{U}^{(e)T} [K^{(e)}] \vec{U}^{(e)}\quad (3)$$

where

$$\vec{U}^{(e)} = \begin{Bmatrix} U_1^{(e)} \\ U_2^{(e)} \end{Bmatrix} = \text{vector of nodal displacements of element}$$

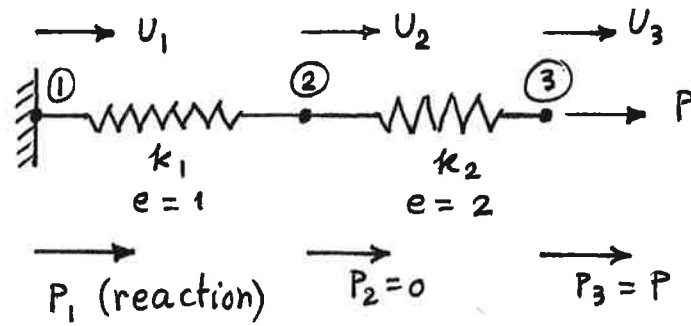
and

$$[K^{(e)}] = \text{stiffness matrix of element.}$$

By comparing Eqs. (2) and (3), the stiffness matrix of the element can be identified as

$$[K^{(e)}] = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\quad (4)$$

1.13



Element stiffness matrices:

$$[k^{(1)}] = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix} \text{ N/m} \quad (1)$$

$$[k^{(2)}] = k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 5 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix} \text{ N/m} \quad (2)$$

Assembled stiffness matrix of the system:

$$[K] = 10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+5 & -5 \\ 0 & -5 & 5 \end{bmatrix} = 10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -5 \\ 0 & -5 & 5 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \end{matrix} \quad (3)$$

Equilibrium equations of the system:

$$10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -5 \\ 0 & -5 & 5 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 1000 \end{Bmatrix} \quad (4)$$

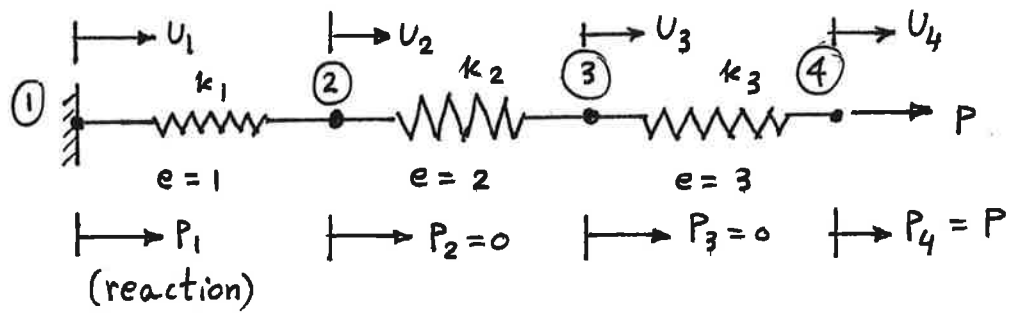
By using the condition  $U_1 = 0$  in Eq. (4), i.e., by deleting the row and column corresponding to  $U_1$  in Eq. (4), we obtain

$$10^5 \begin{bmatrix} 6 & -5 \\ -5 & 5 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} \quad (5)$$

Solution of Eq. (5) gives the nodal displacements:

$$U_2 = 0.010 \text{ m}, \quad U_3 = 0.012 \text{ m} \quad (6)$$

1.14



Element matrices:

$$[k^{(1)}] = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix} \quad \text{N/m}$$

$$[k^{(2)}] = k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix} \quad \text{N/m}$$

$$[k^{(3)}] = k_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 3 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_3 \\ U_4 \end{matrix} \quad \text{N/m}$$

Assembled stiffness matrix of the system:

$$[K] = 10^5 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+2 & -2 & 0 \\ 0 & -2 & 2+3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{matrix} \quad \text{N/m}$$

Equilibrium equations of the system:

$$10^5 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 0 \\ 1000 \end{Bmatrix} \quad (1)$$

By incorporating the boundary condition  $U_1 = 0$ , Eq. (1) reduces to

$$10^5 \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1000 \end{Bmatrix} \quad (2)$$

Solution of Eq. (2):  $U_2 = 10^{-2} \text{ m}$ ,  $U_3 = 1.5 \times 10^{-2} \text{ m}$ ,  $U_4 = 1.8333 \times 10^{-2} \text{ m}$ .