Solutions Manual

The Finite Element Method in Engineering

Fifth Edition

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Solution of all chapter available:

Preface

This solutions manual is intended to aid the instructor in assigning homework problems from the book, Singiresu S. Rao, *The Finite Element Method in Engineering*, 5^{th} Edition, Elsevier Butterworth-Heinemann, Boston, 2011. Solutions of most of the problems are given in this solutions manual. It is advised that the instructor review the problems and solutions before assigning them so that he/she can see the possible alternate approaches for solving a particular problem. Most of the problems can be modified either by changing the data and/or by expanding the scope of the solution to generate new problems. The MATLAB programs described in Chapter 23 are available at the web site of the book: www.elsevierdirect.com/9781856176613. The author appreciates receiving comments on the book as well as any errors found in the book or solutions manual. They can be sent by email to srao@miami.edu. Finally, I would like to thank my wife, Kamala, for her tolerance and patience while preparing this solutions manual.

Singiresu S. Rao

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Chapter 1

Overview of Finite Element Method

1.1 $S^{(\ell)} = n\,r \quad \text{with} \quad r = 2R\sin\left(\frac{1}{2}\cdot\frac{2\pi}{n}\right) = 2R\sin\frac{\pi}{n}$

$S^{(u)} = n\,s \quad \text{with} \quad s = 2R\tan\left(\frac{1}{2}\cdot\frac{2\pi}{n}\right) = 2R\tan\frac{\pi}{n}$

Using series expansions of $\sin\theta$ and $\tan\theta$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \cdots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \cdots$$

we find

$$S^{(\ell)} = 2nR\left(\frac{\pi}{n} - \frac{1}{6}\frac{\pi^3}{n^3} + - \cdots\right)$$

$$= 2\pi R - \frac{\pi^3 R}{3n^2} + - \cdots$$

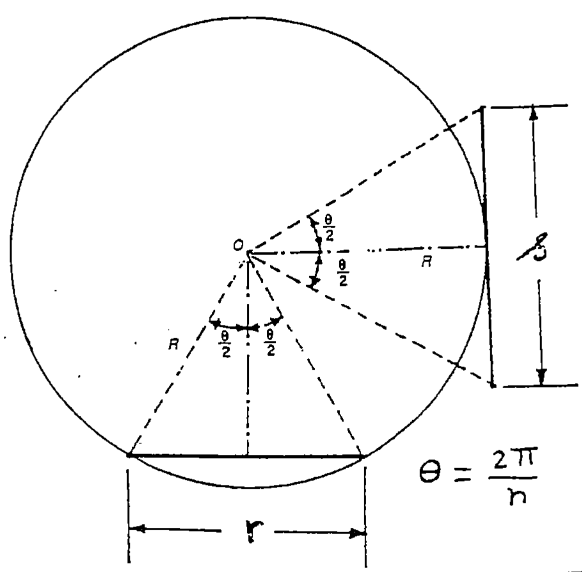
and

$$S^{(u)} = 2nR\left(\frac{\pi}{n} + \frac{1}{3}\frac{\pi^3}{n^3} + \cdots\right)$$

$$= 2\pi R + \frac{2}{3}\frac{\pi^3 R}{n^2} + \cdots$$

Since the true value of S is given by $2\pi R$, we have

$$S^{(\ell)} \leq S \leq S^{(u)}$$



$\theta = \frac{2\pi}{n}$

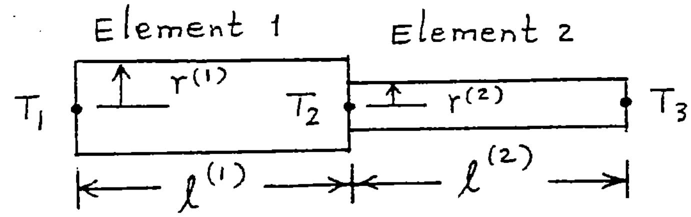
1.2

For $R = 1$:

n	$S^{(l)} = 2nR \sin \frac{\pi}{n}$	$S^{(u)} = 2nR \tan \frac{\pi}{n}$
3	5.1962	10.3923
4	5.6569	8.0000
5	5.8779	7.2654
6	6.0000	6.9282
7	6.0744	6.7420
8	6.1229	6.6274
9	6.1564	6.5515
10	6.1803	6.4984
11	6.1981	6.4598
12	6.2117	6.4308

1.3

1. Idealization $\Rightarrow$



2. Interpolation model:

$$T(x) = a + bx \quad \text{with} \quad a = T_1^{(e)} \quad \text{and} \quad b = \frac{T_2^{(e)} - T_1^{(e)}}{\ell^{(e)}}$$

$$T(x) = T_1^{(e)} + \frac{T_2^{(e)} - T_1^{(e)}}{\ell^{(e)}}\, x$$

3. Element characteristic matrices and vectors:

$$[K^{(e)}] = \frac{1}{\ell^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\, p^{(e)}\, \ell^{(e)}}{6\, k\, A^{(e)}}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

where $p^{(e)} = 2\pi r^{(e)}$ and $A^{(e)} = \pi {r^{(e)}}^2$

$$[K^{(e)}] = \frac{1}{\ell^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\, \ell^{(e)}}{3\, k\, r^{(e)}}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[K^{(1)}] = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{(10)(2)}{3(70)(1)}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6905 & -0.4048 \\ -0.4048 & 0.6905 \end{bmatrix}$$

$$[K^{(2)}] = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{(10)(3)}{3(70)(0.5)}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9048 & -0.04761 \\ -0.04761 & 0.9048 \end{bmatrix}$$

$$\vec{P}^{(e)} = \frac{h\, p^{(e)}\, T_\infty\, \ell^{(e)}}{2\, k\, A^{(e)}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \frac{h\, T_\infty\, \ell^{(e)}}{k\, r^{(e)}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\vec{P}^{(1)} = \frac{(10)(40)(2)}{7(1)}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 11.4286 \\ 11.4286 \end{Bmatrix}$$

$$\vec{P}^{(2)} = \frac{(10)(40)(3)}{70\,(0.5)} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 34.2857 \\ 34.2857 \end{Bmatrix}$$

4. Assembled equations:

$$[\underset{\sim}{K}]\, \underset{\sim}{\vec{T}} = \underset{\sim}{\vec{P}} \tag{1}$$

with

$$[\underset{\sim}{K}] = \begin{bmatrix} 0.6905 & -0.4048 & 0 \\ -0.4048 & 0.6905 + 0.9048 & -0.04761 \\ 0 & -0.04761 & 0.9048 \end{bmatrix}$$

$$\underset{\sim}{\vec{T}} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} \quad \text{and} \quad \underset{\sim}{\vec{P}} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} 11.4286 \\ (11.4286 + 34.2857) \\ 34.2857 \end{Bmatrix} = \begin{Bmatrix} 11.4286 \\ 45.7143 \\ 34.2857 \end{Bmatrix}$$

5. Application of boundary conditions & solution:

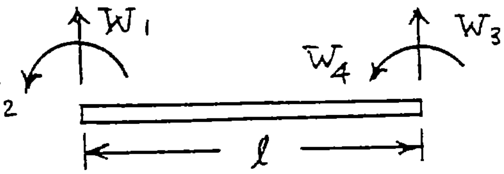
Rewrite Eqs. (1) by taking the terms involving $T_1 = 140$ to the right side (in second and third equations) to obtain

$$\begin{bmatrix} 1.5953 & -0.04761 \\ -0.04761 & 0.9048 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 45.7143 + 0.4048\,(140) \\ 34.2857 \end{Bmatrix}$$

These equations yield

$$T_2 = 65.414\,^\circ C \quad \text{and} \quad T_3 = 41.335\,^\circ C$$

1.4 $w(x) = W_1^{(e)} \cdot \frac{1}{\ell^3}(2x^3 - 3\ell x^2 + \ell^3)$



$$+ W_2^{(e)} \cdot \frac{1}{\ell^2}(x^3 - 2\ell x^2 + \ell^2 x)$$

$$+ W_3^{(e)} \cdot \frac{1}{\ell^3}(3\ell x^2 - 2x^3) + W_4^{(e)} \cdot \frac{1}{\ell^2}(x^3 - \ell x^2)$$

$$[K^{(e)}] = \frac{2EI}{\ell^3}\begin{bmatrix} 6 & 3\ell & -6 & 3\ell \\ 3\ell & 2\ell^2 & -3\ell & \ell^2 \\ -6 & -3\ell & 6 & -3\ell \\ 3\ell & \ell^2 & -3\ell & 2\ell^2 \end{bmatrix} = [\underset{\sim}{K}]$$

$$\vec{W}^{(e)} = \begin{Bmatrix} W_1^{(e)} \\ W_2^{(e)} \\ W_3^{(e)} \\ W_4^{(e)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ W_3 \\ W_4 \end{Bmatrix} = \underset{\sim}{\vec{W}}$$

$$\vec{P}^{(e)} = \begin{Bmatrix} P_1^{(e)} \\ P_2^{(e)} \\ P_3^{(e)} \\ P_4^{(e)} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P \\ 0 \end{Bmatrix} = \underset{\sim}{\vec{P}}$$

$[\underset{\sim}{K}]\underset{\sim}{\vec{W}} = \underset{\sim}{\vec{P}}$ gives

$$\frac{2EI}{\ell^3}\begin{bmatrix} 6 & 3\ell & -6 & 3\ell \\ 3\ell & 2\ell^2 & -3\ell & \ell^2 \\ -6 & -3\ell & 6 & -3\ell \\ 3\ell & \ell^2 & -3\ell & 2\ell^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P \\ 0 \end{Bmatrix} \qquad (1)$$

The third and fourth equations in (1) are given by

$$\frac{2EI}{l^3}\left(6\,W_3 - 3l\,W_4\right) = P \qquad (2)$$

$$\frac{2EI}{l^3}\left(-3l\,W_3 + 2l^2\,W_4\right) = 0 \qquad (3)$$

Solution of Eqs. (2) and (3) is:

$$W_3 = \frac{P l^3}{3EI}, \qquad W_4 = \frac{P l^2}{2EI}$$

The deflection within the element can be expressed as

$$w(x) = \frac{P}{3EI}\left(3l x^2 - 2x^3\right) + \frac{P}{2EI}\left(x^3 - l x^2\right)$$

$$\frac{d^2 w}{dx^2}(x) = \frac{P}{3EI}\left(6l - 12x\right) + \frac{P}{2EI}\left(6x - 2l\right) = \frac{P}{EI}(l - x)$$

Bending moment $= M(x) = EI\,\frac{d^2 w}{dx^2} = P(l - x)$

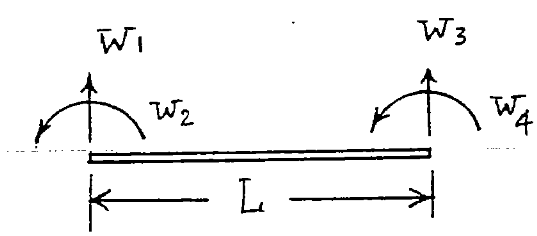
stress in the element $= \sigma(x) = \frac{M(x)\cdot y}{I}$

where y = distance from neutral axis.

$$\therefore\ \sigma(x) = \frac{P}{I}(l - x)\cdot y \quad ; \quad l \equiv L$$

1.5

$$[K^{(e)}] = [\underset{\sim}{K}] = \frac{2EI}{L^3}\begin{array}{c}\left[\begin{array}{cccc} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{array}\right] \begin{array}{c} W_1 \\ W_2 \\ W_3 \\ W_4 \end{array} \\ \begin{array}{cccc} W_1 & W_2 & W_3 & W_4 \end{array}\end{array}$$



$[K]\vec{W} = \vec{P}$ where $[K] = \frac{2EI}{L^3}\begin{bmatrix} 6 & -3L \\ -3L & 2L^2 \end{bmatrix}$

with $\vec{W} = \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix}$ and $\vec{P} = \begin{Bmatrix} 0 \\ M_0 \end{Bmatrix}$

These equations can be rewritten as

$$6\,W_3 - 3L\,W_4 = 0\,, \qquad -3L\,W_3 + 2L^2\,W_4 = \frac{M_o L^3}{2EI}$$

Solution: $\quad W_3 = \dfrac{M_o L^2}{2EI}\,, \quad W_4 = \dfrac{M_o L}{EI}$

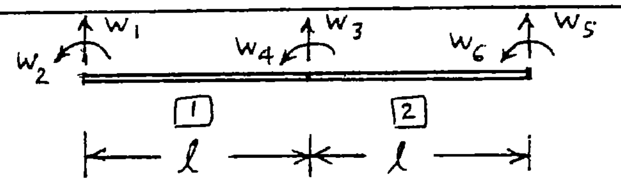
Bending moment in element $= M(x) = EI\,\dfrac{d^2 w}{dx^2}$

with $w(x) = W_3 \cdot \dfrac{1}{L^3}\left(3Lx^2 - 2x^3\right) + W_4 \cdot \dfrac{1}{L^2}\left(x^3 - Lx^2\right)$

and
$$\frac{d^2 w}{dx^2}(x) = W_3 \cdot \frac{1}{L^3}\left(6L - 12x\right) + \frac{W_4}{L^2}\left(6x - 2L\right)$$
$$= \frac{M_o L^2}{2EI}\cdot\frac{1}{L^3}\left(6L - 12x\right) + \frac{M_o L}{EI}\cdot\frac{1}{L^2}\left(6x - 2L\right)$$
$$= \frac{M_o}{EI}$$

$$\sigma(x) = \text{stress} = \frac{M(x)\cdot y}{I} = \left(EI\,\frac{d^2 w}{dx^2}\right)\frac{y}{I} = \frac{M_o \cdot y}{I}$$

1.6 Assembling the element stiffness matrices by retaining only the non-zero degrees of freedom, we obtain



$w_1 = w_2 = w_5 = 0$

$$[K] = \frac{2EI}{\ell^3}\begin{array}{c} \begin{array}{ccc} w_3 & w_4 & w_6 \end{array} \\ \begin{bmatrix} 6+6 & -3\ell+3\ell & 3\ell \\ -3\ell+3\ell & 2\ell^2+2\ell^2 & \ell^2 \\ 3\ell & \ell^2 & 2\ell^2 \end{bmatrix} \end{array} \begin{array}{c} w_3 \\ w_4 \\ w_6 \end{array} \;;\quad \ell = \frac{L}{2}$$

$$= \frac{16EI}{L^3}\begin{bmatrix} 12 & 0 & \frac{3}{2}L \\ 0 & L^2 & \frac{1}{4}L^2 \\ \frac{3}{2}L & \frac{1}{4}L^2 & \frac{1}{2}L^2 \end{bmatrix}$$

Equilibrium equations:

$$\frac{4EI}{L^3}\begin{bmatrix} 48 & 0 & 6L \\ 0 & 4L^2 & L^2 \\ 6L & L^2 & 2L^2 \end{bmatrix}\begin{Bmatrix} w_3 \\ w_4 \\ w_6 \end{Bmatrix} = \begin{Bmatrix} \mathbf{P} \\ 0 \\ 0 \end{Bmatrix} \qquad (1)$$

Scalar form of Eq. (1):

$$48\, W_3 + 6L\, W_6 = \frac{PL^3}{4EI} \tag{2}$$

$$4L^2 W_4 + L^2 W_6 = 0 \quad \Rightarrow \quad W_4 = -\frac{W_6}{4} \tag{3}$$

$$6L\, W_3 + L^2 W_4 + 2L^2 W_6 = 0 \tag{4}$$

Eqs. (3) and (4) give

$$W_3 = -\frac{L^2}{6L} W_4 - \frac{2L^2}{6L} W_6 = -\frac{7}{24} L\, W_6 \tag{5}$$

Eqs. (2), (5) and (3) yield

$$W_3 = \frac{7}{768}\frac{PL^3}{EI}, \quad W_4 = \frac{1}{128}\frac{PL^2}{EI}, \quad W_6 = -\frac{PL^2}{32EI} \tag{6}$$

Bending moment in element $= M(x) = EI \dfrac{d^2 w}{dx^2}$

$$\text{stress in element} = \sigma(x) = \frac{M(x)\cdot y}{I} = E\,y\,\frac{d^2 w}{dx^2} \tag{7}$$

$$\sigma_{xx}^{(1)} = -E\,y\left[\frac{1}{L^3}(6L - 12x)\frac{7PL^3}{768EI} + \frac{1}{L^2}(6x - 2L)\frac{PL^2}{128EI}\right]$$

$$= -y\,\frac{P}{I}\left(\frac{5L - 8x}{128}\right) \tag{7}$$

$$\sigma_{xx}^{(2)} = -y\,E\left[\frac{1}{L^3}(12x - 6L)\frac{7PL^3}{768EI} + \frac{1}{L^2}(6x - 4L)\frac{PL^2}{128EI}\right.$$

$$\left. + \frac{1}{L^2}(6x - 2L)\left(-\frac{PL^2}{32EI}\right)\right]$$

$$= y\,\frac{P}{I}\left(\frac{x + 2L}{32}\right) \tag{8}$$

1.7 Assembled stiffness matrix, [K], is same as the one derived in Problem 1.6. Equilibrium equations:

$$\frac{4EI}{L^3}\begin{bmatrix} 48 & 0 & 6L \\ 0 & 4L^2 & L^2 \\ 6L & L^2 & 2L^2 \end{bmatrix}\begin{Bmatrix} W_3 \\ W_4 \\ W_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_0 \\ 0 \end{Bmatrix} \qquad (1)$$

Solution of Eqs. (1):

$$W_3 = \frac{M_0 L^2}{128\,EI}, \quad W_4 = \frac{5\,M_0 L}{64\,EI}, \quad W_6 = -\frac{M_0 L}{16\,EI}$$

stress element 1:

$$\sigma(x) = -Ey\frac{d^2w}{dx^2} = -Ey\left[\frac{W_3}{\ell^3}(6\ell - 12x) + \frac{W_4}{\ell^2}(6x - 2\ell)\right]$$

$$= -Ey\left[\frac{M_0 L^2}{128\,EI}\cdot\frac{8}{L^3}(3L - 12x) + \frac{5M_0L}{64EI}\cdot\frac{4}{L^2}(6x - L)\right]$$

$$= -\frac{M_0 y}{8IL}(9x - L)$$

Stress in element 2:

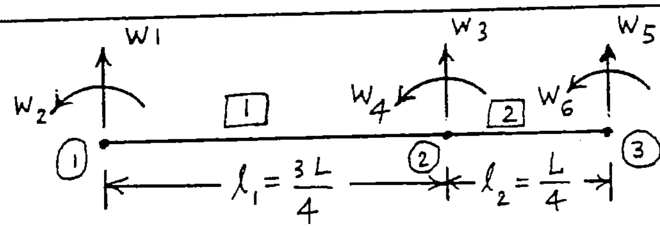
$$\sigma(x) = -Ey\left[\frac{W_3}{\ell^3}(12x - 6\ell) + \frac{W_4}{\ell^2}(6x - 4\ell) + \frac{W_6}{\ell^2}(6x - 2\ell)\right]$$

$$= -Ey\left[\frac{M_0 L^2}{128EI}\cdot\frac{8}{L^3}(12x - 3L) + \frac{5M_0L}{64EI}\cdot\frac{4}{L^2}(6x - 2L) - \frac{M_0 L}{16EI}\cdot\frac{4}{L^2}\cdot(6x - L)\right]$$

$$= -\frac{9M_0 y}{16IL}(2x - L)$$

1.8

2 element idealization ⇒



$$[K^{(e)}] = \frac{2EI}{\ell_e^3}\begin{bmatrix} 6 & 3\ell_e & -6 & 3\ell_e \\ 3\ell_e & 2\ell_e^2 & -3\ell_e & \ell_e^2 \\ -6 & -3\ell_e & 6 & -3\ell_e \\ 3\ell_e & \ell_e^2 & -3\ell_e & 2\ell_e^2 \end{bmatrix}$$

where ℓ_e = length of element e.

Boundary conditions: $W_1 = W_2 = W_5 = W_6 = 0$.

Considering only the free degrees of freedom, the assembled stiffness matrix of the system can be obtained as

$$[K] = 2EI \begin{array}{c} \begin{array}{cc} W_3 & W_4 \end{array} \\ \left[\begin{array}{c|c} \frac{6}{\ell_1^3} + \frac{6}{\ell_2^3} & -\frac{3}{\ell_1^2} + \frac{3}{\ell_2^2} \\ \hline -\frac{3}{\ell_1^2} + \frac{3}{\ell_2^2} & \frac{2}{\ell_1} + \frac{2}{\ell_2} \end{array}\right] \end{array} \begin{array}{c} W_3 \\ W_4 \end{array} = \begin{bmatrix} \left(\frac{7168}{9}\frac{EI}{L^3}\right) & \left(\frac{256}{3}\frac{EI}{L^2}\right) \\ \left(\frac{256}{3}\frac{EI}{L^2}\right) & \left(\frac{64}{3}\frac{EI}{L}\right) \end{bmatrix}$$

Equilibrium equations: $[K]\vec{W} = \vec{P}$

i.e.

$$\begin{bmatrix} \frac{7168}{9}\frac{EI}{L^3} & \frac{256}{3}\frac{EI}{L^2} \\ \frac{256}{3}\frac{EI}{L^2} & \frac{64}{3}\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Solution is:

$$W_3 = \frac{9}{4096}\frac{PL^3}{EI}, \quad W_4 = -\frac{9}{1024}\frac{PL^2}{EI}$$

Stress in element 1:

$$w^{(1)}(x) = \frac{9}{4096}\frac{PL^3}{EI}\cdot\frac{1}{l_1^3}\left(3l_1 x^2 - 2x^3\right) - \frac{9}{1024}\frac{PL^2}{EI} *$$

$$\frac{1}{l_1^2}\left(x^3 - l_1 x^2\right)$$

$$= \frac{P}{192\,EI}\left(\frac{9}{2}Lx^2 - 5x^3\right)$$

$$\sigma^{(1)}(x) = \frac{M^{(1)}y}{I} = E\,y\,\frac{d^2w^{(1)}}{dx^2} = E\,y\cdot\frac{P}{192\,EI}\left(9L - 30x\right)$$

$$= \frac{P}{64\,I}\left(3L - 10x\right)y.$$

Stress in element 2:

$$w^{(2)}(x) = \frac{9}{4096}\frac{PL^3}{EI}\left(\frac{1}{l_2^3}\right)\left(2x^3 - 3l_2x^2 + l_2^3\right)$$

$$-\frac{9}{1024}\frac{PL^2}{EI}\left(\frac{1}{l_2^2}\right)\left(x^3 - 2l_2x^2 + l_2^2x\right)$$

$$= \frac{9}{64}\frac{P}{EI}\left(x^3 - \frac{L}{4}x^2 - \frac{L^2}{16}x + \frac{L^3}{64}\right)$$

$$\sigma^{(2)}(x) = \frac{M^{(2)}y}{I} = E\,y\,\frac{d^2w^{(2)}}{dx^2} = E\,y\cdot\frac{9P}{128EI}\left(12x - L\right)$$

$$= \frac{9P}{128\,I}\left(12x - L\right)y.$$

1.9 $A(x) = A_0\, e^{-x/l}$

$$\Pi = \text{strain energy} = \frac{1}{2}\int_0^l \sigma\,\epsilon\, A(x)\cdot dx$$

$$= \frac{1}{2}\int_0^l E\left(\frac{\partial u}{\partial x}\right)^2 A(x)\, dx \quad \text{with} \quad u(x) = \left(1 - \frac{x}{l}\right)U_1 + \left(\frac{x}{l}\right)U_2$$

$$\Pi = \frac{1}{2}\int_0^l E\left(\frac{U_2 - U_1}{l}\right)^2 A_0\, e^{-x/l}\, dx$$

$$= \frac{1}{2} E A_0 \left(\frac{U_2 - U_1}{l}\right)^2 \int_0^l e^{-x/l}\, dx$$

$$= \frac{1}{2} E A_0 \left(\frac{U_2 - U_1}{l}\right)^2 (0.6321\, l) = \frac{1}{2}\{U_1 \;\; U_2\}[K]\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}$$

with

$$[K] = \frac{0.6321\, E A_0}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$T = \text{kinetic energy} = \frac{1}{2}\int_0^l \rho\left(\frac{\partial u}{\partial t}\right)^2 A(x)\cdot dx$$

$$= \frac{1}{2}\int_0^l \{\dot{U}_1 \;\; \dot{U}_2\}\begin{bmatrix} \left(1 - \frac{x}{l}\right)^2 & \left(1 - \frac{x}{l}\right)\frac{x}{l} \\ \left(1 - \frac{x}{l}\right)\frac{x}{l} & \left(\frac{x}{l}\right)^2 \end{bmatrix}\begin{Bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{Bmatrix} A_0\, e^{-x/l}\cdot dx$$

$$= \frac{1}{2}\dot{\vec{U}}^T [M]\,\dot{\vec{U}}$$

Using $\int_0^l e^{-x/l}\cdot dx = 0.6321\, l$,

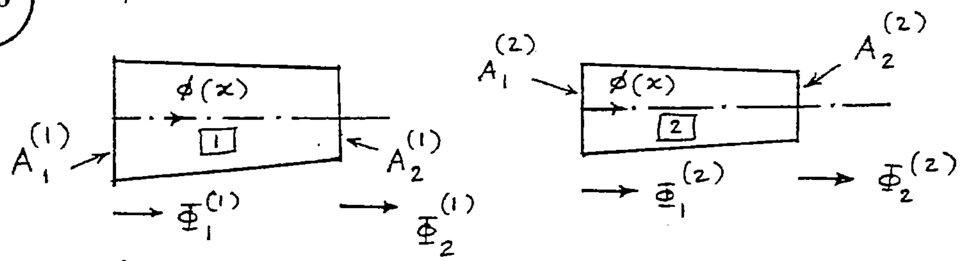
$\int_0^l \frac{x}{l}\, e^{-x/l}\cdot dx = 0.2642\, l$,

and $\int_0^l \frac{x^2}{l^2}\, e^{-x/l}\cdot dx = 0.1605\, l$, we obtain

$$[M] = \rho A_0 l \begin{bmatrix} 0.2642 & 0.1037 \\ 0.1037 & 0.1605 \end{bmatrix}$$

where ρ = density and l = length of element.

1.10 Step 1. Idealization:



$\ell^{(i)} = 5$ cm, $E^{(i)} = 2 \times 10^7$ N/cm^2 ; $i = 1, 2$

$A_1^{(1)} = 2$ cm^2, $A_2^{(1)} = A_1^{(2)} = 1.5$ cm^2, $A_2^{(2)} = 1$ cm^2

Step 2. Interpolation model:

$$\phi(x) = \left[\left(1 - \frac{x}{\ell^{(e)}}\right) \quad \left(\frac{x}{\ell^{(e)}}\right)\right] \begin{Bmatrix} \Phi_1^{(e)} \\ \Phi_2^{(e)} \end{Bmatrix}$$

Step 3. Element equations:

strain energy of element e = $\int_0^{\ell^{(e)}} \left(\frac{1}{2}\sigma^{(e)}\epsilon^{(e)}\right) A^{(e)}\,dx$

with $\sigma^{(e)} = E^{(e)}\epsilon^{(e)}$, $\epsilon^{(e)} = \frac{d\phi}{dx} = \frac{\Phi_2^{(e)} - \Phi_1^{(e)}}{\ell^{(e)}}$

Here $A^{(e)}(x) = A_1^{(e)} + \frac{A_2^{(e)} - A_1^{(e)}}{\ell^{(e)}}\, x$

Hence

$$\pi^{(e)} = \text{strain energy} = \frac{E^{(e)}}{2}\left(\frac{\Phi_2^{(e)} - \Phi_1^{(e)}}{\ell^{(e)}}\right)^2 \int_0^{\ell^{(e)}} \left\{A_1^{(e)} + \frac{A_2^{(e)} - A_1^{(e)}}{\ell^{(e)}}\, x\right\} dx$$

$$= \frac{A_{av}^{(e)} E^{(e)}}{2\,\ell^{(e)}} \left(\Phi_1^{(e)^2} + \Phi_2^{(e)^2} - 2\,\Phi_1^{(e)}\Phi_2^{(e)}\right)$$

where $A_{av}^{(e)}$ = average area = $\frac{A_1^{(e)} + A_2^{(e)}}{2}$

Expressing $\pi^{(e)}$ as $\pi^{(e)} = \frac{1}{2}\,\vec{\Phi}^{(e)^T}[K^{(e)}]\,\vec{\Phi}^{(e)}$,

we obtain

$$[K^{(e)}] = \frac{A_{av}^{(e)} E^{(e)}}{\ell^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{with} \quad \vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1^{(e)} \\ \Phi_2^{(e)} \end{Bmatrix}$$

$$A_{av}^{(1)} = \frac{2+1.5}{2} = 1.75 \text{ cm}^2, \quad A_{av}^{(2)} = \frac{1.5+1}{2} = 1.25 \text{ cm}^2$$

$$[K^{(1)}] = 10^6 \begin{bmatrix} 7 & -7 \\ -7 & 7 \end{bmatrix}, \quad [K^{(2)}] = 10^6 \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$$

Step 4. Assembly of equations:

$$[\underset{\sim}{K}] = 10^6 \begin{bmatrix} 7 & -7 & 0 \\ -7 & 7+5 & -5 \\ 0 & -5 & 5 \end{bmatrix}$$

$$\vec{\underset{\sim}{\Phi}} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix}, \quad \vec{\underset{\sim}{P}} = \begin{Bmatrix} P_1 \\ 0 \\ 1 \end{Bmatrix}, \quad P_1 = \text{reaction at node } 1$$

$$[\underset{\sim}{K}]\, \vec{\underset{\sim}{\Phi}} = \vec{\underset{\sim}{P}} \qquad (1)$$

Step 5. Apply boundary conditions and solve equations:

Since $\Phi_1 = 0$, we delete first row and first column in Eqs. (1) to obtain

$$[K]\, \vec{\Phi} = \vec{P} \quad \Rightarrow \quad 10^6 \begin{bmatrix} 12 & -5 \\ -5 & 5 \end{bmatrix} \begin{Bmatrix} \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \qquad (2)$$

Solution of Eqs. (2) gives

$$\Phi_2 = 1.428 \times 10^{-7} \text{ cm}, \quad \Phi_3 = 3.428 \times 10^{-7} \text{ cm}$$

Step 6. Find stresses:

$$\sigma^{(1)} = E^{(1)} \epsilon^{(1)} = E^{(1)} \frac{\Phi_2^{(1)} - \Phi_1^{(1)}}{l^{(1)}} = (2 \times 10^7)\left(\frac{1.428 \times 10^{-7} - 0}{5}\right)$$

$$= 0.5712 \text{ N/cm}^2$$

$$\sigma^{(2)} = E^{(2)} \epsilon^{(2)} = E^{(2)} \frac{\Phi_2^{(2)} - \Phi_1^{(2)}}{l^{(2)}}$$

$$= (2 \times 10^7)\left(\frac{3.428 \times 10^{-7} - 1.428 \times 10^{-7}}{5}\right) = 0.8 \text{ N/cm}^2$$

1.11 (a) Element characteristic matrices:

$$c_1 = \frac{\pi d_1^4}{128 \mu_1 \ell_1} = \frac{\pi \; (5^4)}{128 \, (1{\cdot}6 \times 10^{-6})(1000)} = 9.5874 \times 10^3 \ \text{in}^5/\text{lb-sec}$$

$$c_2 = \frac{\pi d_2^4}{128 \mu_2 \ell_2} = \frac{\pi \; (2^4)}{128 \, (1{\cdot}6 \times 10^{-6})(1500)} = 0.1636 \times 10^3 \ \text{in}^5/\text{lb-sec}$$

$$c_3 = \frac{\pi d_3^4}{128 \mu_3 \ell_3} = \frac{\pi \; (4^4)}{128 \, (1{\cdot}6 \times 10^{-6})(2000)} = 1.9635 \times 10^3 \ \text{in}^5/\text{lb-sec}$$

Pipe element 1 with nodes 1 and 2:

$$[K^{(1)}] = c_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 9587.4 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

Pipe element 2 with nodes 2 and 3:

$$[K^{(2)}] = c_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1636.0 \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \end{array}$$

Pipe element 3 with nodes 2 and 4:

$$[K^{(3)}] = c_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1963.5 \begin{array}{c} \begin{array}{cc} 2 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 4 \end{array}$$

Assembled characteristic matrix:

$$[\underset{\sim}{K}] = \begin{bmatrix} 9587.4 & -9587.4 & 0 & 0 \\ -9587.4 & \begin{matrix}(9587.4 \\ +163.6 \\ +1963.5)\end{matrix} & -163.6 & -1963.5 \\ 0 & -163.6 & 163.6 & 0 \\ 0 & -1963.5 & 0 & 1963.5 \end{bmatrix}$$

System equations:

$$\begin{bmatrix} 9587.4 & -9587.4 & 0 & 0 \\ -9587.4 & 11714.5 & -163.6 & -1963.5 \\ 0 & -163.6 & 163.6 & 0 \\ 0 & -1963.5 & 0 & 1963.5 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} \quad (1)$$

where Q_i is the external flow rate at node i.

Since the pressures at nodes 1, 3 and 4 are known, p_2 is the unknown pressure in Eq. (1) with $Q_2 = 0$. By using the known values, $p_1 = 20$, $p_3 = 15$ and $p_4 = 15$ psi, Eq. (1) can be rewritten in the form

$$\begin{bmatrix} 9587.4 & 0 & 0 & 0 \\ 0 & 11714.5 & 0 & 0 \\ 0 & 0 & 163.6 & 0 \\ 0 & 0 & 0 & 1963.5 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 9587.4 * 20 \leftarrow p_1 \\ 9587.4\ p_1 + 163.6\ p_3 + 1963.5\ p_4 = 223655 \\ 163.6 * 15 \leftarrow p_3 \\ 1963.5 * 15 \leftarrow p_4 \end{Bmatrix} \quad (2)$$

Eq. (2) gives the nodal pressures as:

$$p_1 = 20 \text{ psi}, \quad p_2 = \frac{223655}{11714.5} = 19.0921 \text{ psi},$$

$$p_3 = 15 \text{ psi}, \quad p_4 = 15 \text{ psi}$$

(b) Volume flow rates in pipe elements:

$$Q_1 = c_1 (p_1 - p_2) = 9587.4 (20 - 19.0921)$$
$$= 8704.323 \text{ in}^3/\text{sec}$$

$$Q_2 = c_2 (p_1 - p_2) = 1636.0 (19.0921 - 15)$$
$$= 669.4676 \text{ in}^3/\text{sec}$$

$$Q_3 = c_3 (p_2 - p_4) = 1963.5 (19.0921 - 15)$$
$$= 8034.838 \ \text{in}^3/\text{sec}$$

It can be verified that the inflow rate (Q_1) is equal to the total outflow rate $(Q_2 + Q_3)$.

(c) Reynolds number for flow in each of the pipe elements :

$$R_e = \frac{\rho v d}{\mu} = \frac{\rho Q d}{\left(\frac{\pi d^2}{4}\right)\mu} = \frac{4 \rho Q}{\pi d \mu}$$

For pipe element 1:

$$R_e = 4 \left(\frac{1}{12}\right)^4 \frac{\rho Q}{\pi \mu d} = 4 \left(\frac{1}{12}\right)^4 \frac{(1.9)(8704.3)}{\pi (1.6 \times 10^{-6})(5)}$$
$$= 0.126936 \times 10^6$$

For pipe element 2:

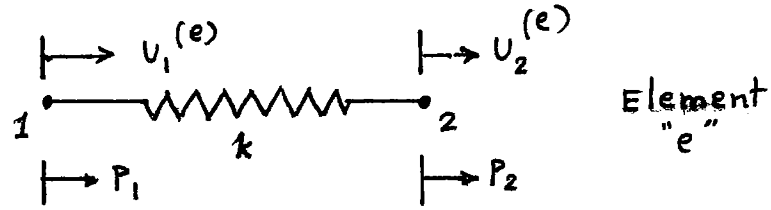
$$R_e = 4 \left(\frac{1}{12}\right)^4 \frac{(1.9)(669.5)}{\pi (1.6 \times 10^{-6})(2)} = 0.024411 \times 10^6$$

For pipe element 3:

$$R_e = 4 \left(\frac{1}{12}\right)^4 \frac{(1.9)(8034.8)}{\pi (1.6 \times 10^{-6})(4)} = 0.146465 \times 10^6$$

(d) Since $R_e > 2000$ in each pipe elements (segments), the flow is turbulent in all pipe elements.

1.12



When nodal forces P_1 and P_2 act along the nodal displacements $u_1^{(e)}$ and $u_2^{(e)}$, respectively, the strain energy of the spring is given by

$$\pi = \frac{1}{2}\,(\text{net force})\,(\text{net displacement})$$

$$= \frac{1}{2}\left\{k\left(u_2^{(e)} - u_1^{(e)}\right)\right\}\left(u_2^{(e)} - u_1^{(e)}\right)$$

$$= \frac{1}{2}\,k\,\left(u_2^{(e)} - u_1^{(e)}\right)^2 \tag{1}$$

Eq. (1) can be expressed in matrix form as

$$\pi = \frac{1}{2}\left\{u_1^{(e)} \quad u_2^{(e)}\right\}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix} \tag{2}$$

or

$$\pi = \frac{1}{2}\,\vec{U}^{(e)T}\,[K^{(e)}]\,\vec{U}^{(e)} \tag{3}$$

where

$\vec{U}^{(e)} = \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix}$ = vector of nodal displacements of element

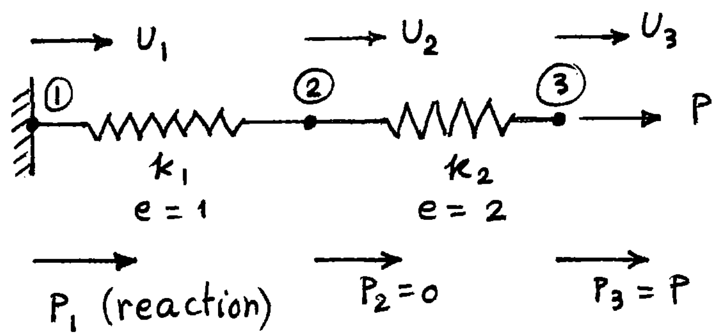
and

$[K^{(e)}]$ = stiffness matrix of element.

By comparing Eqs. (2) and (3), the stiffness matrix of the element can be identified as

$$[K^{(e)}] = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{4}$$

1.13



Element stiffness matrices:

$$[k^{(1)}] = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{c} \begin{array}{cc} U_1 & U_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \end{array} \quad N/m \qquad (1)$$

$$[k^{(2)}] = k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 5 \times 10^5 \begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array} \quad N/m \qquad (2)$$

Assembled stiffness matrix of the system :

$$[\underset{\sim}{K}] = 10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+5 & -5 \\ 0 & -5 & 5 \end{bmatrix} = 10^5 \begin{array}{c} \begin{array}{ccc} U_1 & U_2 & U_3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -5 \\ 0 & -5 & 5 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \\ U_3 \end{array} \qquad (3)$$

Equilibrium equations of the system:

$$10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -5 \\ 0 & -5 & 5 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 1000 \end{Bmatrix} \qquad (4)$$

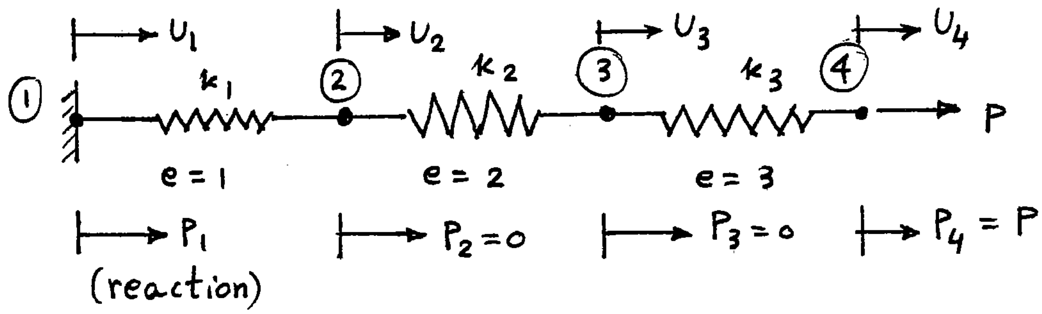
By using the condition $U_1 = 0$ in Eq. (4), i.e., by deleting the row and column corresponding to U_1 in Eq. (4), we obtain

$$10^5 \begin{bmatrix} 6 & -5 \\ -5 & 5 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} \qquad (5)$$

Solution of Eq. (5) gives the nodal displacements:

$$U_2 = 0.010 \text{ m}, \quad U_3 = 0.012 \text{ m} \qquad (6)$$

1.14



Element matrices:

$$\left[k^{(1)}\right] = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{c} \begin{array}{cc} U_1 & U_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \end{array} \quad N/m$$

$$\left[k^{(2)}\right] = k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2 \times 10^5 \begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array} \quad N/m$$

$$\left[k^{(3)}\right] = k_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 3 \times 10^5 \begin{array}{c} \begin{array}{cc} U_3 & U_4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_3 \\ U_4 \end{array} \quad N/m$$

Assembled stiffness matrix of the system:

$$\left[\underset{\sim}{K}\right] = 10^5 \begin{array}{c} \begin{array}{cccc} U_1 & U_2 & U_3 & U_4 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+2 & -2 & 0 \\ 0 & -2 & 2+3 & -3 \\ 0 & 0 & -3 & 0 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \\ U_3 \\ U_4 \end{array} \quad N/m$$

Equilibrium equations of the system:

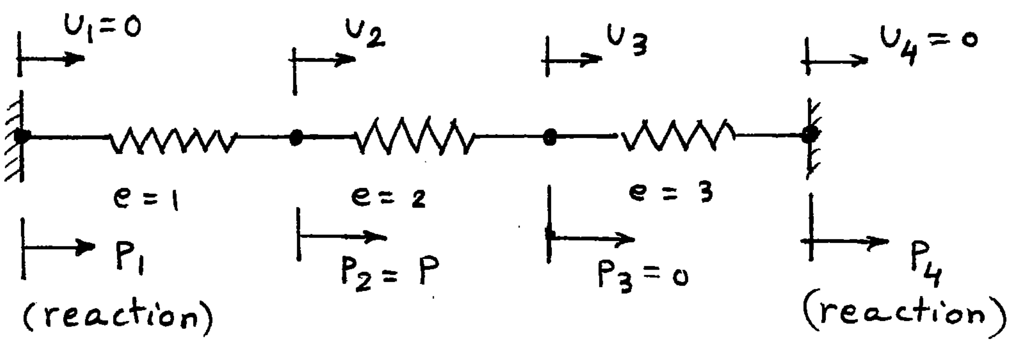
$$10^5 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 0 \\ 1000 \end{Bmatrix} \qquad (1)$$

By incorporating the boundary condition $U_1 = 0$, Eq. (1) reduces to

$$10^5 \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1000 \end{Bmatrix} \qquad (2)$$

Solution of Eq. (2): $U_2 = 10^{-2}$ m, $U_3 = 1.5 \times 10^{-2}$ m, $U_4 = 1.8333 \times 10^{-2}$ m.

1.15



Element matrices and assembled stiffness matrix of the system are same as the ones given in the Solution of Problem 1·14.

System equilibrium equations (before applying the known boundary conditions):

$$10^5 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P = 1000 \\ 0 \\ P_4 \end{Bmatrix} \qquad (1)$$

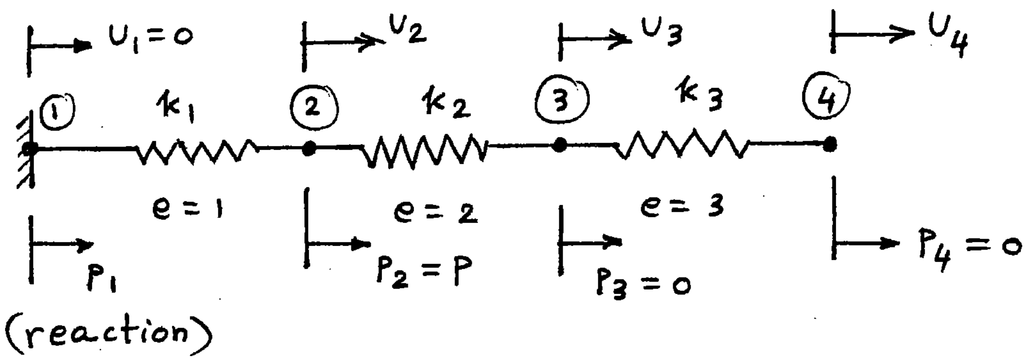
Using the known boundary conditions, $U_1 = U_4 = 0$, Eq. (1) reduces to

$$10^5 \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 0 \end{Bmatrix} \qquad (2)$$

solution of Eq. (2) is:

$$U_2 = 0.004545 \text{ m}, \quad U_3 = 0.001818 \text{ m} \qquad (3)$$

1.16



Element stiffness matrices are same as the ones given in the Solution of Problem 1.14.

System equations (before applying the known boundary condition):

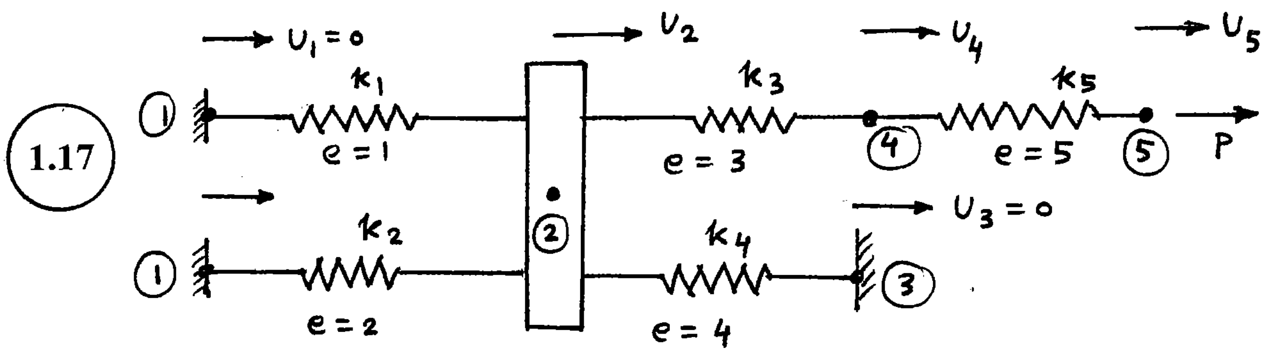
$$10^5 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 1000 \\ 0 \\ 0 \end{Bmatrix} \qquad (1)$$

System equations (after using the boundary condition, $U_1 = 0$):

$$10^5 \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 0 \\ 0 \end{Bmatrix} \qquad (2)$$

Solution of Eq. (2):

$$\vec{U} = \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{Bmatrix} 10^{-2} \text{ m} \qquad (3)$$



Element stiffness matrices:

$$[k^{(1)}] = 10^5 \begin{array}{c} \begin{array}{cc} U_1 & U_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \end{array} \quad N/m$$

$$[k^{(2)}] = 2 \times 10^5 \begin{array}{c} \begin{array}{cc} U_1 & U_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \end{array} \quad N/m$$

$$[k^{(3)}] = 3 \times 10^5 \begin{array}{c} \begin{array}{cc} U_2 & U_4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_4 \end{array} \quad N/m$$

$$[k^{(4)}] = 4 \times 10^5 \begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array} \quad N/m$$

$$[k^{(5)}] = 5 \times 10^5 \begin{array}{c} \begin{array}{cc} U_4 & U_5 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_4 \\ U_5 \end{array} \quad N/m$$

Assembled stiffness matrix of the system:

$$[\underset{\sim}{K}] = 10^5 \begin{array}{c} \begin{array}{ccccc} U_1 & U_2 & U_3 & U_4 & U_5 \end{array} \\ \begin{bmatrix} 1+2 & -1-2 & 0 & 0 & 0 \\ -1-2 & 1+2+3+4 & -4 & -3 & 0 \\ 0 & -4 & 4 & 0 & 0 \\ 0 & -3 & 0 & 3+5 & -5 \\ 0 & 0 & 0 & -5 & 5 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{array} \quad N/m$$

system equilibrium equations (before applying the boundary conditions):

$$10^5 \begin{bmatrix} 3 & -3 & 0 & 0 & 0 \\ -3 & 10 & -4 & -3 & 0 \\ 0 & -4 & 4 & 0 & 0 \\ 0 & -3 & 0 & 8 & -5 \\ 0 & 0 & 0 & -5 & 5 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ P_3 \\ 0 \\ P_5 = 1000 \end{Bmatrix}$$

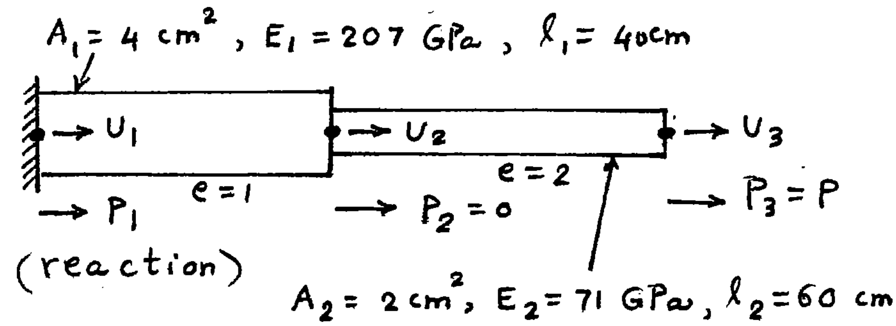
system equations (after using the known boundary conditions, $U_1 = U_3 = 0$):

$$10^5 \begin{bmatrix} 10 & -3 & 0 \\ -3 & 8 & -5 \\ 0 & -5 & 5 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1000 \end{Bmatrix} \qquad (1)$$

Solution of Eqs. (1):

$$\vec{U} = \begin{Bmatrix} U_2 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 0.001428 \\ 0.004762 \\ 0.006762 \end{Bmatrix} \text{ m} \qquad (2)$$

1.18



Element stiffness matrices:

$$[K^{(1)}] = \frac{A_1 E_1}{\ell_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(4\times 10^{-4})(207\times 10^{9})}{40\times 10^{-2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 20.7 \times 10^{7} \begin{array}{c} \begin{array}{cc} U_1 & U_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ U_1 \\ U_2 \end{array} \quad N/m$$

$$[K^{(2)}] = \frac{A_2 E_2}{\ell_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(2\times 10^{-4})(71\times 10^{9})}{60\times 10^{-2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 2.3667 \times 10^{7} \begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ U_2 \\ U_3 \end{array} \quad N/m$$

Assembled stiffness matrix:

$$[\underset{\sim}{K}] = 10^{7}\begin{bmatrix} 20.7 & -20.7 & 0 \\ -20.7 & 20.7+2.3667 & -2.3667 \\ 0 & -2.3667 & 2.3667 \end{bmatrix} \quad N/m$$

system equilibrium equations before applying boundary conditions:

$$10^{7}\begin{bmatrix} 20.7 & -20.7 & 0 \\ -20.7 & 23.0667 & -2.3667 \\ 0 & -2.3667 & 2.3667 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 1000 \end{Bmatrix} \quad (1)$$

system equilibrium equations after applying boundary conditions

$$10^{7}\begin{bmatrix} 23.0667 & -2.3667 \\ -2.3667 & 2.3667 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} \quad (2)$$

Solution of Eq. (2):

$$\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = 10^{-4} \begin{Bmatrix} 0.0483 \\ 0.4708 \end{Bmatrix} \text{ m}$$

stress in Element 1:

$$\sigma_1 = E_1 \epsilon_1 = E_1 \left(\frac{U_2 - U_1}{l_1} \right)$$

$$= 207 \times 10^9 \left(\frac{0.0483 - 0}{40 \times 10^{-2}} \right) \times 10^{-4}$$

$$= 2.4995 \times 10^6 \text{ N/m}^2$$

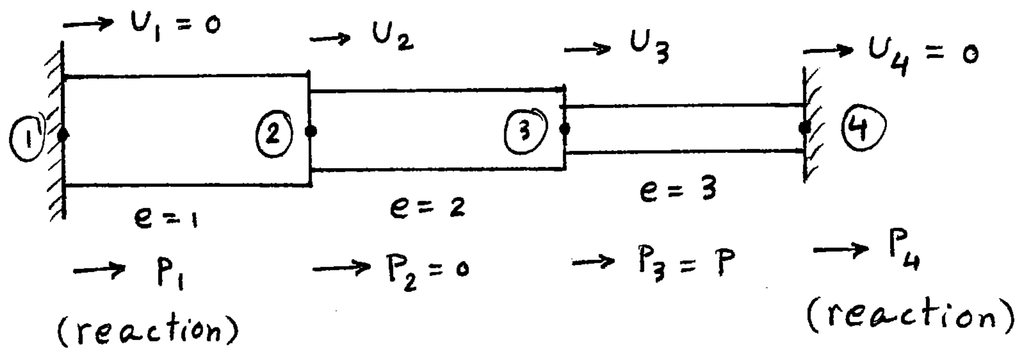
stress in element 2:

$$\sigma_2 = E_2 \epsilon_2 = E_2 \left(\frac{U_3 - U_2}{l_2} \right)$$

$$= 71 \times 10^9 \left(\frac{0.4708 - 0.0483}{60 \times 10^{-2}} \right) 10^{-4}$$

$$= 4.9996 \times 10^6 \text{ N/m}^2$$

1.19



Element stiffness matrices:

$A_1 = 5 \times 10^{-4}\ m^2$, $E_1 = 207 \times 10^9$ Pa, $l_1 = 0.2$ m

$$\frac{A_1 E_1}{l_1} = \frac{(5 \times 10^{-4})(207 \times 10^9)}{0.2} = 517.5 \times 10^6 \ N/m$$

$$[K^{(1)}] = 517.5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} N/m$$

$$\frac{A_2 E_2}{l_2} = \frac{(3 \times 10^{-4})(207 \times 10^9)}{0.3} = 207.0 \times 10^6 \ N/m$$

$$[K^{(2)}] = 207.0 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} N/m$$

$A_3 = 1 \times 10^{-4}\ m^2$, $E_3 = 207 \times 10^9$ Pa, $l_3 = 0.4$ m

$$\frac{A_3 E_3}{l_3} = \frac{(1 \times 10^{-4})(207 \times 10^9)}{0.4} = 51.75 \times 10^6 \ N/m$$

$$[K^{(3)}] = 51.75 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} N/m$$

Assembled stiffness matrix:

$$[\underset{\sim}{K}] = 10^6 \begin{bmatrix} 517.5 & -517.5 & 0 & 0 \\ -517.5 & 517.5 + 207.0 & -207.0 & 0 \\ 0 & -207.0 & 207.0 + 51.75 & -51.75 \\ 0 & 0 & -51.75 & 51.75 \end{bmatrix} N/m$$

System equilibrium before applying boundary conditions:

$$10^6 \begin{bmatrix} 517.5 & -517.5 & 0 & 0 \\ -517.5 & 724.5 & -207.0 & 0 \\ 0 & -207.0 & 258.75 & -51.75 \\ 0 & 0 & -51.75 & 51.75 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 2000 \\ P_4 \end{Bmatrix}$$

System equilibrium equations after incorporating the boundary conditions $U_1 = U_4 = 0$ (by deleting rows and columns 1 and 4 in above equations):

$$10^6 \begin{bmatrix} 724.5 & -207.0 \\ -207.0 & 258.75 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2000 \end{Bmatrix} \qquad (1)$$

Solution of Eq. (1):

$$\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = 10^{-4} \begin{Bmatrix} 0.0286 \\ 0.1002 \end{Bmatrix} \text{ m}$$

Stress in element (step) 1:

$$\sigma_1 = E_1 \epsilon_1 = E_1 \left(\frac{U_2 - U_1}{l_1} \right) = 207 \times 10^9 \left(\frac{0.0286 - 0}{0.2} \right) 10^{-4}$$

$$= 2.9601 \times 10^6 \text{ N/m}^2$$

Stress in element (step) 2:

$$\sigma_2 = E_2 \epsilon_2 = E_2 \left(\frac{U_3 - U_2}{l_2} \right) = 207 \times 10^9 \left(\frac{0.1002 - 0.0286}{0.3} \right) 10^{-4}$$

$$= 4.9404 \times 10^6 \text{ N/m}^2$$

Stress in element (step) 3:

$$\sigma_3 = E_3 \epsilon_3 = E_3 \left(\frac{U_4 - U_3}{l_3} \right) = 207 \times 10^9 \left(\frac{0 - 0.1002}{0.4} \right) 10^{-4}$$

$$= -5.1853 \times 10^6 \text{ N/m}^2$$

1.20 ① idealization : divide the frame into n beam elements :

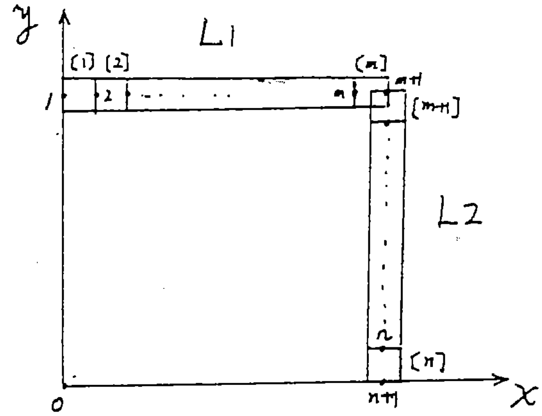


Fig. (a)

② interpolation model and element matrices

i) for elements in L_1 section , shown in Fig. (a),

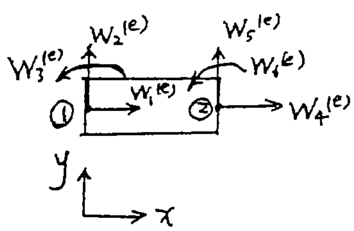


Fig. 1

two uniaxial displacements exist : $u_{x_1}^{(e)}$, $u_{y_1}^{(e)}$

$$\begin{cases} u_{x_1}^{(e)} = a_1 + a_2 x \\ u_{y_1}^{(e)} = b_0 + b_1 x + b_2 x^2 + b_3 x^3 \end{cases}$$

where a_1, a_2 are determined by $W_1^{(e)}$, $W_4^{(e)}$;

b_0, b_1, b_2, b_3 are determined by $W_2^{(e)}$, $W_3^{(e)}$, $W_5^{(e)}$, $W_6^{(e)}$.

strain energy of the element :

$$\pi_{L_1}^{(e)} = \frac{1}{2}\int_0^{l^{(e)}} \left[E^{(e)} \cdot \left(\frac{du_{x_1}}{dx}\right)^2 \cdot A^{(e)} + E^{(e)} I^{(e)} \cdot \left(\frac{d^2 u_y}{dx^2}\right)^2 \right] \cdot dx$$

$$= \frac{1}{2} \vec{W}^{(e)T}_{1\times 6} \left[K^{(e)} \right]_{6\times 6} \cdot \vec{W}^{(e)}_{6\times 1}$$

$$\vec{W}^{(e)} = \begin{bmatrix} W_1^{(e)} \\ W_2^{(e)} \\ W_3^{(e)} \\ W_4^{(e)} \\ W_5^{(e)} \\ W_6^{(e)} \end{bmatrix}$$

ii) for elements in L2 section, shown in Fig. (b),

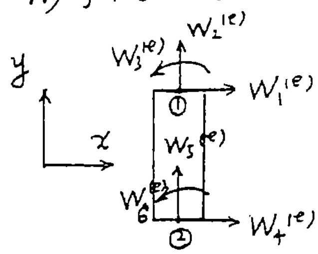


Fig. (b)

two uniaxial displacements exist : $U_{x_2}^{(e)}$, $U_{y_2}^{(e)}$

$$\begin{cases} U_{x_2}^{(e)} = c_0 + c_1 y + c_2 y^2 + c_3 y^3 \\ U_{y_2}^{(e)} = d_0 + d_1 y \end{cases}$$

where d_0, d_1 are ditermined by $W_2^{(e)}$, $W_5^{(e)}$;

c_0, c_1, c_2, c_3 are ditermined by $W_1^{(e)}, W_3^{(e)}, W_4^{(e)}, W_6^{(e)}$

strain energy of this element :

$$\pi_{L_2}^{(e)} = \frac{1}{2}\int_0^{l^{(e)}} \left[EI^{(e)} \left(\frac{d^2 U_{x_2}^{(e)}}{dy^2} \right)^2 + E^{(e)} \left(\frac{dU_{y_2}}{dy} \right)^2 \cdot A^{(e)} \right] \cdot dy$$

$$= \frac{1}{2} \vec{W}_{1\times 6}^{(e)T} \cdot [K^{(e)}]_{6\times 6} \cdot \vec{W}_{6\times 1}^{(e)}$$

$$\vec{W}_{1\times 6}^{(e)} = \begin{bmatrix} W_1^{(e)} \\ W_2^{(e)} \\ W_3^{(e)} \\ W_4^{(e)} \\ W_5^{(e)} \\ W_6^{(e)} \end{bmatrix}$$

③ assembly of element and derivation of system equation

$$I = \sum_{i=1}^{m} \pi_{L_1}^{(e)} + \sum_{i=m+1}^{n} \pi_{L_2}^{(e)} - W_p$$

$$W_p = P_1 \cdot W_{3m+2} + P_2 \cdot W_{3m+1}$$

$$\frac{\partial I}{\partial W_i} = 0, \quad i = 1, 2, \cdots, 3n,$$

we can get system equation :

$$[\underset{\sim}{K}]_{3n\times 3n} \cdot \vec{\underline{W}}_{3n\times 1} = \vec{\underline{P}}_{3n\times 1}$$

where $\underline{\vec{W}}^T = [W_1, W_2, \cdots, W_{3n}]$

considering B.C. : $W_1 = W_2 = W_3 = 0$, $W_{3n+2} = W_{3n+2} = W_{3n+3} = 0$:

the system equation can be simplied :

$$[K]_{(3n-6)\times(3n-6)} \cdot \vec{W}_{(3n+6)\times 1} = \vec{P}_{(3n-6)\times 1}$$

solve this equations, $\vec{W}$ can be obtained, so the displacement interpolation functions $u_{x_1}^{(e)}$, $u_{y_1}^{(e)}$, $u_{x_2}^{(e)}$, $u_{y_2}^{(e)}$ will be determined.

④ stress distribution :

i) For L1 section :

$$\sigma_x^{(e)} = E \cdot \epsilon_x^{(e)} = E \cdot \left(\frac{du_{x_1}}{dx} - y \cdot \frac{d^2 u_{y_1}}{dx^2} \right)$$

ii) For L2 section :

$$\sigma_y^{(e)} = E \cdot \epsilon_y^{(e)} = E \cdot \left(\frac{du_{y_2}}{dy} - x \cdot \frac{d^2 u_{x_2}}{dy^2} \right)$$

1.21

Eq. (1.18) :

$$\pi^{(e)} = \frac{1}{2} \int_0^{l^{(e)}} E^{(e)} I^{(e)} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx \qquad (E.1)$$

where $w(x)$ is given by Eq. (1.17) :

$$w(x) = N_1(x) \cdot W_1^{(e)} + N_2(x) \, W_2^{(e)} + N_3(x) \cdot W_3^{(e)} + N_4(x) \, W_4^{(e)} \qquad (E.2)$$

Differentiation of Eq. (E.2) twice with respect to x yields

$$\frac{\partial^2 w}{\partial x^2} = N_1'' \, W_1^{(e)} + N_2'' \, W_2^{(e)} + N_3'' \, W_3^{(e)} + N_4'' \, W_4^{(e)} \qquad (E.3)$$

with

$$N_1''(x) = -\frac{6}{l^{(e)^2}} + 12 \frac{x}{l^{(e)^3}}, \quad N_2''(x) = -\frac{4}{l^{(e)}} + 6 \frac{x}{l^{(e)^2}},$$

$$N_3''(x) = \frac{6}{l^{(e)^2}} - 12 \frac{x}{l^{(e)^3}}, \quad N_4''(x) = -\frac{2}{l^{(e)}} + 6 \frac{x}{l^{(e)^2}} \qquad (E.4)$$

Substitution of Eqs. (E.4) and (E.3) into Eq. (E.1) leads to terms of the type

$$k_{ij} = E^{(e)} I^{(e)} \int_0^{l^{(e)}} N_i''(x) \, N_j''(x) \, dx \qquad (E.5)$$

The integral in Eq. (E.5) can be evaluated to find the stiffness coefficients k_{ij}. For example,

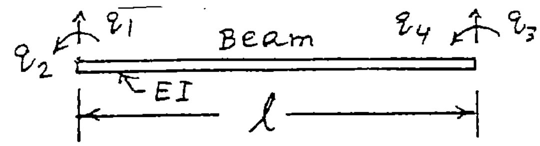
$$k_{11} = E^{(e)} I^{(e)} \int_0^{l^{(e)}} \left(-\frac{6}{l^{(e)^2}} + 12 \frac{x}{l^{(e)^3}} \right)^2 dx$$

$$= \frac{E^{(e)} I^{(e)}}{l^{(e)^4}} \left\{ 36 x - \frac{72 x^2}{l^{(e)}} + 48 \frac{x^3}{l^{(e)^2}} \right\}_0^{l^{(e)}}$$

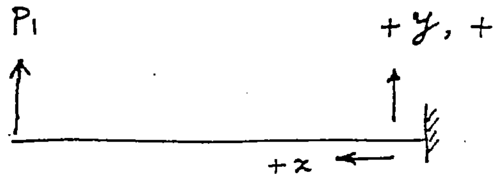
$$= 12 \, E^{(e)} I^{(e)} / l^{(e)^3} \qquad (E.6)$$

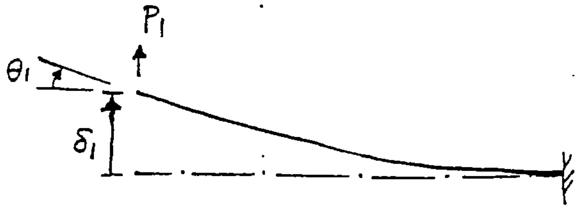
where $E^{(e)} I^{(e)}$ is assumed to be constant. Evaluation of integrals in Eq. (E.5) for $i, j = 1$ to 4 leads to Eq. (1.19).

1.22 Beam deflection formulas:

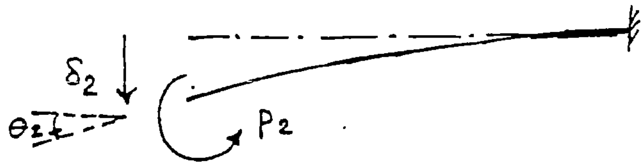


Degrees of freedom





$$\delta_1 = \frac{P_1 l^3}{3EI}, \quad \theta_1 = \frac{P_1 l^2}{2EI}$$



$$\delta_2 = -\frac{P_2 l^2}{2EI}$$

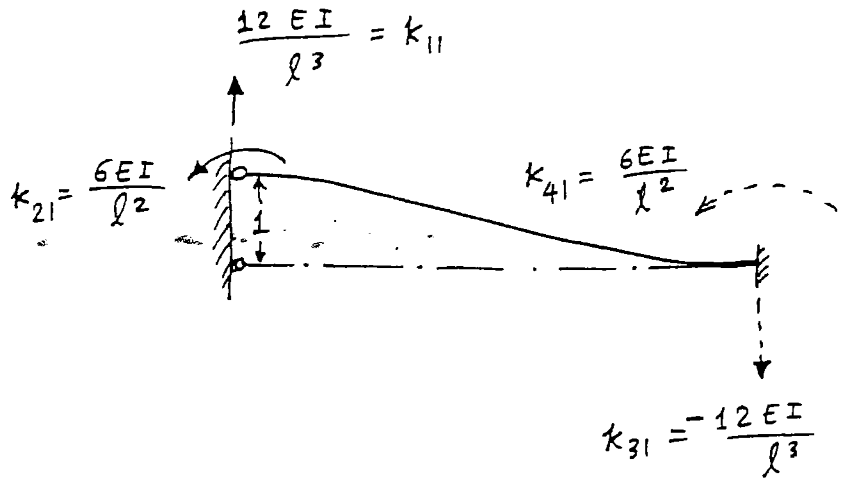
$$\theta_2 = -\frac{P_2 l}{EI}$$

Apply P_1 and P_2 such that

$$\delta_1 + \delta_2 = 1$$

$$\theta_1 + \theta_2 = 0$$

$$\left.\begin{aligned} P_1\left(\frac{l^3}{3EI}\right) - P_2\left(\frac{l^2}{2EI}\right) &= 1 \\ P_1\left(\frac{l^2}{2EI}\right) - P_2\left(\frac{l}{EI}\right) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} P_1 &= \frac{12EI}{l^3} = k_{11} \\ P_2 &= \frac{6EI}{l^2} = k_{21} \end{aligned}$$



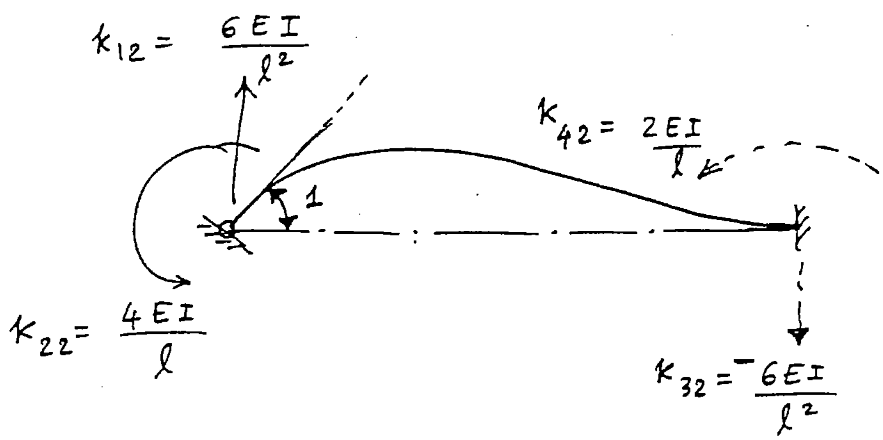
Reactions at right end can be found from equilibrium equations.

1.23 Apply P_1 and P_2 such that

$$\delta_1 + \delta_2 = 0$$

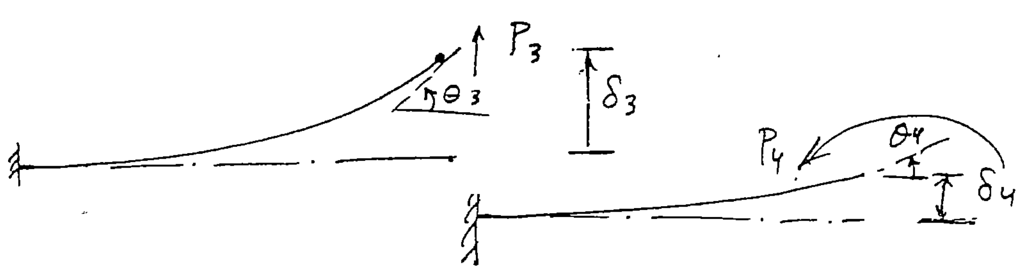
$$\theta_1 + \theta_2 = -1$$

$$\left.\begin{aligned} P_1\left(\frac{\ell^3}{3EI}\right) - P_2\left(\frac{\ell^2}{2EI}\right) &= 0 \\ P_1\left(\frac{\ell^2}{2EI}\right) - P_2\left(\frac{\ell}{EI}\right) &= -1 \end{aligned}\right\} \Rightarrow \begin{aligned} P_1 &= \frac{6EI}{\ell^2} = k_{12} \\ P_2 &= \frac{4EI}{\ell} = k_{22} \end{aligned}$$



Reactions at right end can be found from equilibrium equations

1.24



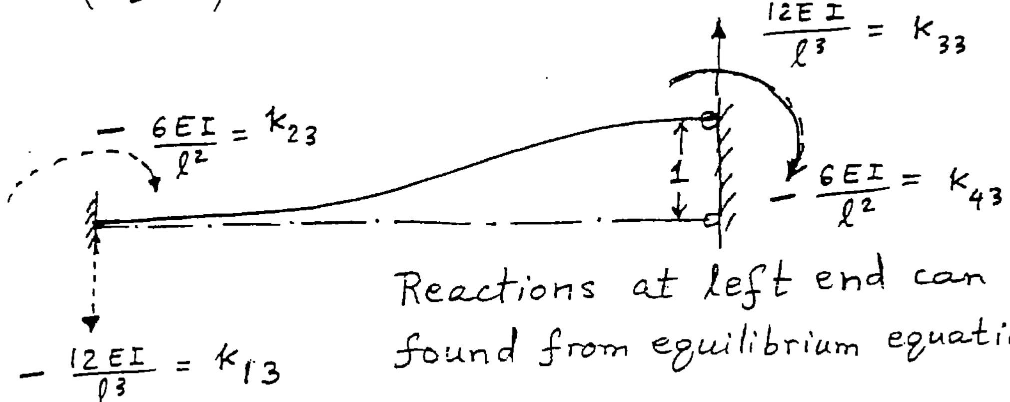
$$\delta_3 = \frac{P_3 \ell^3}{3EI} \quad ; \quad \theta_3 = +\frac{P_3 \ell^2}{2EI}$$

$$\delta_4 = +\frac{P_4 \ell^2}{2EI} \quad ; \quad \theta_4 = +\frac{P_4 \ell}{EI}$$

Apply P_3 and P_4 such ~~that~~ $\delta_3 + \delta_4 = 1$

$\theta_3 + \theta_4 = 0$

$$\left.\begin{aligned} P_3\left(\frac{\ell^3}{3EI}\right) + P_4\left(\frac{\ell^2}{2EI}\right)\Big] &= 1 \\ P_3\left(\frac{\ell^2}{2EI}\right) + P_4\left(\frac{\ell}{EI}\right) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} P_3 &= \frac{12EI}{\ell^3} = k_{33} \\ P_4 &= -\frac{6EI}{\ell^2} = k_{43} \end{aligned}$$



Reactions at left end can be found from equilibrium equations

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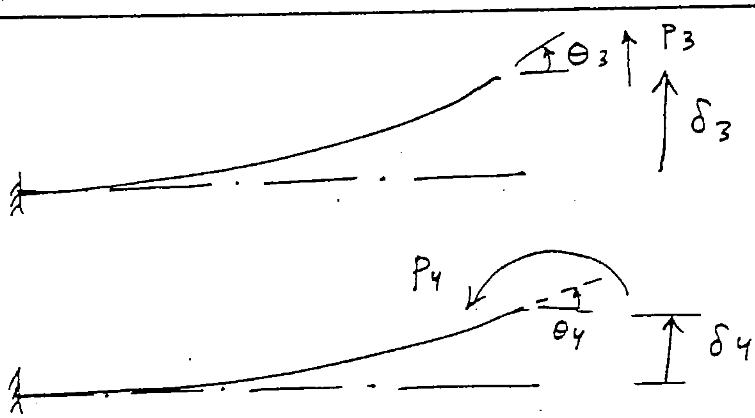
1.25 $\quad 1.25\frac{1}{2}$

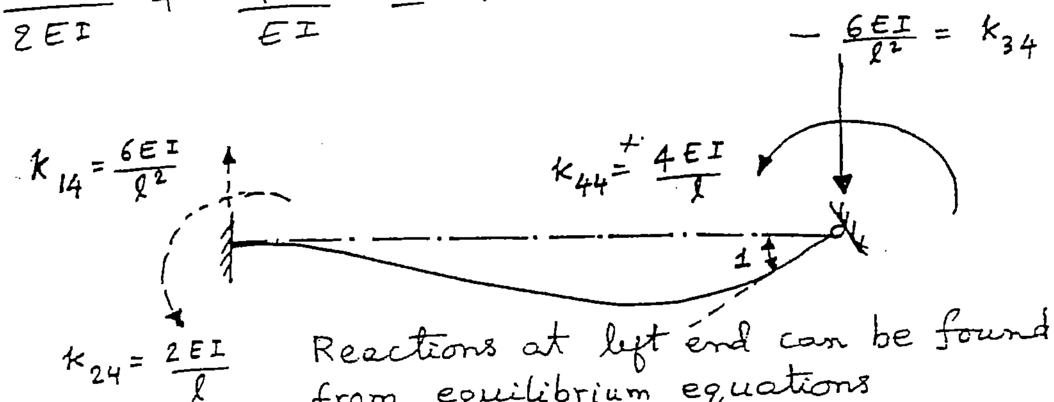


$$\delta_3 = \frac{P_3 \ell^3}{3EI} \quad ; \quad \theta_3 = \frac{P_3 \ell^2}{2EI}$$

$$\delta_4 = \frac{P_4 \ell^2}{2EI} \quad ; \quad \theta_4 = \frac{P_4 \ell}{EI}$$

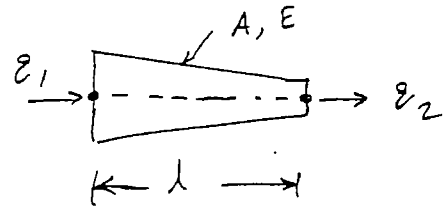
Apply P_3 and P_4 such that $\delta_3 + \delta_4 = 0$
$\theta_3 + \theta_4 = 1$

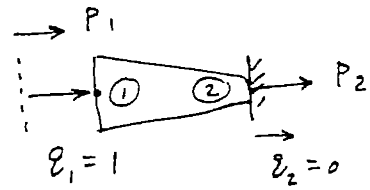
$$\left.\begin{aligned} \frac{P_3 l^3}{3EI} + \frac{P_4 \cdot l^2}{2EI} &= 0 \\ \frac{P_3 l^2}{2EI} + \frac{P_4 l}{EI} &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} P_3 &= -\frac{6EI}{l^2} = k_{34} \\ P_4 &= \frac{4EI}{l} = k_{44} \end{aligned}$$



Reactions at left end can be found from equilibrium equations

1.26

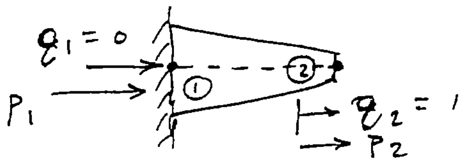




P_1 = force needed to cause $q_1 = 1$ (while $q_2 = 0$) at node 1 = (stress)(area) = (E)(strain)(average area)

$= E \cdot \frac{1}{l}\left(\frac{A_1 + A_2}{2}\right) = k_{11}$

$P_1 + P_2 = 0$ for equilibrium: $P_2 = -P_1 = k_{12}$.



P_2 = force needed to cause $q_2 = 1$ (while $q_1 = 0$) at node 2 = (stress) (area)

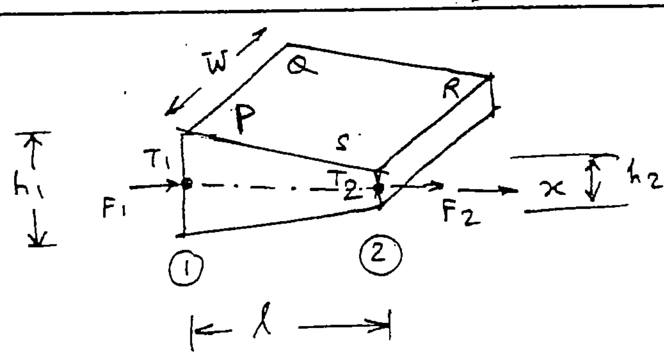
$= (E)(\text{strain})(\text{average area}) = E \cdot \frac{1}{l} \cdot \left(\frac{A_1 + A_2}{2}\right) = k_{22}$

$P_1 + P_2 = 0$ for equilibrium. $P_1 = -P_2 = k_{21}$

$$\therefore [k] = \left(\frac{A_1 + A_2}{2}\right)\frac{E}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

1.27 $1{\cdot}27\frac{1}{2}$

Neglect convection from slant surfaces (such as PQRS).



Heat flowing across unit area at node 1 in x-direction per unit time $= -k_c \frac{dT}{dx}$

where k_c is the thermal conductivity.

$\therefore$ Heat flux entering at node ① $= -k_c A_1 \frac{dT}{dx} = F_1$

Where A_1 = area of cross-section at node ①

$= h_1 W$

Using $\frac{dT}{dx} = \frac{T_2 - T_1}{l}$, F_1 can be expressed as

$$F_1 = -k_c A_1 \cdot \left(\frac{T_2 - T_1}{l} \right) \qquad (a)$$

Similarly, heat flux entering node ② in x-direction is

$$F_2 = -k_c A_2 \left(\frac{T_1 - T_2}{l} \right) \qquad (b)$$

Equations (a) and (b) can be written in matrix form as

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{k_c}{l} \begin{bmatrix} A_1 & -A_1 \\ -A_2 & A_2 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

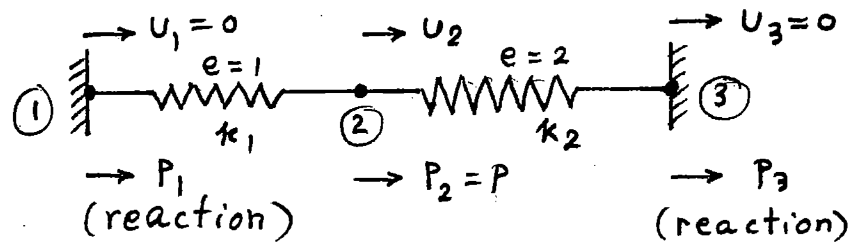
Thus the characteristic matrix of the fin is given by

$$[k] = \frac{k_c}{l} \begin{bmatrix} A_1 & -A_1 \\ -A_2 & A_2 \end{bmatrix}$$

If the average area, $(A_1 + A_2)/2$, is used in place of A_1 and A_2, we obtain

$$[k] = \frac{k_c (A_1 + A_2)}{2l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

1.28



Element stiffness matrices:

$$[k^{(1)}] = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 500 \begin{array}{c} \begin{array}{cc} U_1 & U_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \end{array} \; N/mm \qquad (1)$$

$$[k^{(2)}] = k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 800 \begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array} \; N/mm \qquad (2)$$

Assembled stiffness matrix:

$$[\underset{\sim}{K}] = \begin{bmatrix} 500 & -500 & 0 \\ -500 & 500+800 & -800 \\ 0 & -800 & 800 \end{bmatrix} N/mm \qquad (3)$$

System equilibrium equations (before applying boundary conditions):

$$\begin{bmatrix} 500 & -500 & 0 \\ -500 & 1300 & -800 \\ 0 & -800 & 800 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 1000 \\ P_3 \end{Bmatrix} \qquad (4)$$

System equilibrium equations (after applying boundary conditions, $U_1 = U_3 = 0$):

$$(1300)\, U_2 = 1000 \qquad (5)$$

Solution of Eq. (5):

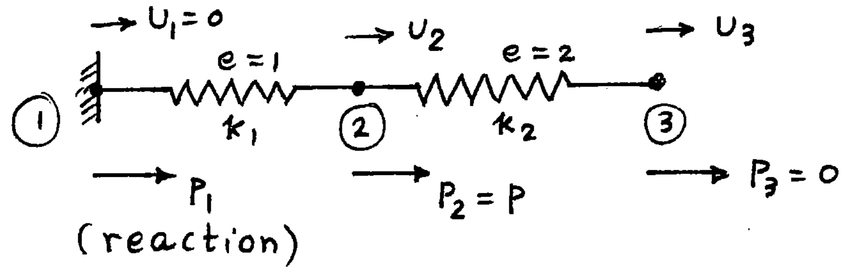
$$U_2 = 0.7692 \text{ mm} \qquad (6)$$

Reactions at fixed ends: (Use Eq. (4)):

$P_1 = 500\, U_1 - 500\, U_2 = -384.6$ N

$P_3 = -800\, U_2 + 800\, U_3 = -615.4$ N

(1.29)



Element stiffness matrices:

$$[K^{(1)}] = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 400 \begin{array}{cc} U_1 & U_2 \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} U_1 \\ U_2 \end{array} \ \text{N/mm} \quad (1)$$

$$[K^{(2)}] = k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 600 \begin{array}{cc} U_2 & U_3 \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} U_2 \\ U_3 \end{array} \ \text{N/mm} \quad (2)$$

Assembled stiffness matrix of the system:

$$[\underset{\sim}{K}] = \begin{bmatrix} 400 & -400 & 0 \\ -400 & 400 + 600 & -600 \\ 0 & -600 & 600 \end{bmatrix} \ \text{N/mm} \quad (3)$$

system equilibrium equations (before applying the boundary condition):

$$\begin{bmatrix} 400 & -400 & 0 \\ -400 & 1000 & -600 \\ 0 & -600 & 600 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 800 \\ 0 \end{Bmatrix} \quad (4)$$

system equilibrium equations, by deleting row and column 1 in Eq. (4) to incorporate the condition, $U_1 = 0$:

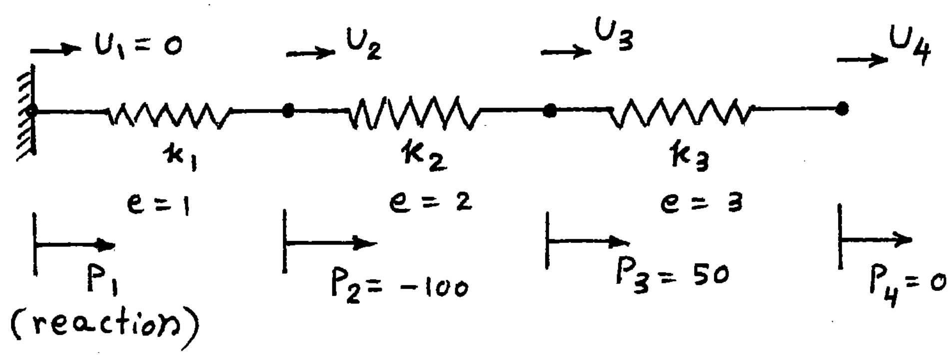
$$\begin{bmatrix} 1000 & -600 \\ -600 & 600 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 800 \\ 0 \end{Bmatrix} \quad (5)$$

solution of Eq. (5): $U_2 = U_3 = 2$ mm

Reaction at fixed node (From row 1 of Eq. (4)):

$$P_1 = 400\, U_1 - 400\, U_2 = -800 \ \text{N} \quad (6)$$

1.30



Element stiffness matrices:

$$[K^{(i)}] = k_i \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{with } k_1 = 40,\ k_2 = 20,\ k_3 = 60\ \frac{N}{mm}$$

Assembled stiffness matrix of the system:

$$[\underset{\sim}{K}] = \begin{bmatrix} 40 & -40 & 0 & 0 \\ -40 & 40+20 & -20 & 0 \\ 0 & -20 & 20+60 & -60 \\ 0 & 0 & -60 & 60 \end{bmatrix} N/mm$$

System equilibrium equations before applying b.c.s:

$$\begin{bmatrix} 40 & -40 & 0 & 0 \\ -40 & 60 & -20 & 0 \\ 0 & -20 & 80 & -60 \\ 0 & 0 & -60 & 60 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ -100 \\ 50 \\ 0 \end{Bmatrix} \qquad (1)$$

System equations after applying the known boundary condition, $U_1 = 0$:

$$\begin{bmatrix} 60 & -20 & 0 \\ -20 & 80 & -60 \\ 0 & -60 & 60 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} -100 \\ 50 \\ 0 \end{Bmatrix} \qquad (2)$$

Solution of Eq. (2):

$U_2 = -1.25$ mm, $U_3 = 1.25$ mm, $U_4 = 1.25$ mm

1.31

$$\mu = 2\times10^{-6}\ \text{lbf - sec/in}^2$$
$$\rho = 2.0\ \text{slugs/ft}^3$$
$$\left[Q^{(e)}\right]=\left[K^{(e)}\right]\vec{p}=\frac{\pi d^{(e)4}}{128\mu l^{(e)}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}p_i^{(e)}\\ p_j^{(e)}\end{Bmatrix}$$

For the pipe network

$$\left[K^{(1)}\right]=\frac{\pi(2)^4}{128(2\times10^{-6})(10000)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=19.6350\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\ \text{for nodes 1 and 3}$$
$$\left[K^{(2)}\right]=\frac{\pi(1.5)^4}{128(2\times10^{-6})(15000)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=4.1417\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\ \text{for nodes 2 and 3}$$
$$\left[K^{(3)}\right]=\frac{\pi(2.5)^4}{128(2\times10^{-6})(20000)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=23.9684\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\ \text{for nodes 3 and 4}$$
$$\left[K^{(4)}\right]=\frac{\pi(3)^4}{128(2\times10^{-6})(25000)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=39.7608\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\ \text{for nodes 3 and 5}$$

Assembled the global matrix is

$$[K]=\begin{bmatrix}19.6350 & 0 & -19.6350 & 0 & 0\\ 0 & 4.1417 & -4.1417 & 0 & 0\\ -19.6350 & -4.1417 & 87.5059 & -23.9684 & -39.7608\\ 0 & 0 & -23.9684 & 23.9684 & 0\\ 0 & 0 & -39.7608 & 0 & 39.7608\end{bmatrix}$$

We know that $\vec{P}=\begin{Bmatrix}p_1\\ p_2\\ p_3\\ p_4\\ p_5\end{Bmatrix}=\begin{Bmatrix}28\\ 20\\ p_3\\ 22\\ 24\end{Bmatrix}$ thus we can write in terms of a diagonal matrix as

$$\begin{Bmatrix}Q_1\\ Q_2\\ Q_3\\ Q_4\\ Q_5\end{Bmatrix}=\begin{bmatrix}19.6350 & 0 & 0 & 0 & 0\\ 0 & 4.1417 & 0 & 0 & 0\\ 0 & 0 & 87.5059 & 0 & 0\\ 0 & 0 & 0 & 23.9684 & 0\\ 0 & 0 & 0 & 0 & 39.7608\end{bmatrix}\begin{Bmatrix}p_1\\ p_2\\ p_3\\ p_4\\ p_5\end{Bmatrix}=\begin{Bmatrix}549.779\\ 82.835\\ 2114.178\\ 527.306\\ 254.259\end{Bmatrix}$$

These equations can be solve to find that

$$p_3 = 24.1604\ \text{psi}$$

a. laminar flow in each pipe segment?

From the definition of Reynolds number

Reynold's number (R_e) is given by

$$R_e = \frac{\rho V d}{\mu} = \frac{2.0}{(144^2)} \frac{V.d}{(2 \times 10^{-6})} = \frac{V d}{0.020736}$$

where

$$V_1 = \frac{Q_1}{A_1} = \frac{75.3925}{3.1416} = 23.9981 \text{ in/sec}$$

$$V_2 = \frac{Q_2}{A_2} = \frac{17.2320}{\frac{\pi}{4}(1.5^2)} = 9.7513 \text{ in/sec}$$

$$V_3 = \frac{51.7824}{\frac{\pi}{4}(2.5^2)} = \frac{Q_3}{A_3} = 10.5490 \text{ in/sec}$$

$$V_4 = \frac{Q_4}{A_4} = \frac{6.3735}{\frac{\pi}{4}(3^2)} = 1.1091 \text{ in/sec}$$

Hence,

For element 1: $R_e = \frac{V_1 d_1}{0.020736} = \frac{23.9981\,(2)}{0.020736} = 2309.0638$ Turbulent

For element 2: $R_e = \frac{V_2 d_2}{0.020736} = \frac{9.7513\,(1.5)}{0.020736} = 705.3891$ Laminar

For element 3: $R_e = \frac{V_3 d_3}{0.020736} = \frac{10.5490\,(2.5)}{0.020736} = 1271.8219$ Laminar

For element 4: $R_e = \frac{V_4 d_4}{0.020736} = \frac{1.1091\,(3.0)}{0.020736} = 160.4601$ Laminar

b. Pressure at node 3

From above we know that $p_3 = 24.1604$ psi

c. Volume flow rates in each pipe segment

$$\begin{Bmatrix} Q^{(1)} \\ Q^{(2)} \\ Q^{(3)} \\ Q^{(4)} \end{Bmatrix} = \begin{Bmatrix} c_1(p_1 - p_2) \\ c_2(p_3 - p_2) \\ c_3(p_3 - p_4) \\ c_4(p_3 - p_5) \end{Bmatrix} = \begin{Bmatrix} 75.3904 \\ 17.2313 \\ 51.7814 \\ 6.37762 \end{Bmatrix} \text{ in}^3/\text{s}$$

As a check , we can see that mass is conserved because total inflow at 1 equals total outflow at 2, 3 and 4.

1.32

$$c_1 = \frac{\pi d_1^4}{128\, \mu_1 \ell_1} = \frac{\pi (3^4)}{128(1.5\times10^{-6})(5000)} = 0.2651 \times 10^3$$

$$c_2 = \frac{\pi d_2^4}{128\, \mu_2 \ell_2} = \frac{\pi (2^4)}{128\,(1.5\times10^{-6})(7000)} = 0.0374 \times 10^3$$

$$c_3 = \frac{\pi d_3^4}{128\, \mu_3 \ell_3} = \frac{\pi (1.5^4)}{128\,(1.5\times10^{-6})(4000)} = 0.1473 \times 10^3$$

$$c_4 = \frac{\pi d_4^4}{128\, \mu_4 \ell_4} = \frac{\pi (2^4)}{128\,(1.5\times10^{-6})(6000)} = 0.0436 \times 10^3$$

$$c_5 = \frac{\pi d_5^4}{128\, \mu_5 \ell_5} = \frac{\pi (2.5^4)}{128\,(1.5\times10^{-6})(8000)} = 0.0799 \times 10^3$$

$$c_6 = \frac{\pi d_6^4}{128\, \mu_6 \ell_6} = \frac{\pi (1^4)}{128\,(1.5\times10^{-6})(5000)} = 0.00327 \times 10^3$$

$$c_7 = \frac{\pi d_7^4}{128\, \mu_7 \ell_7} = \frac{\pi (2^4)}{128\,(1.5\times10^{-6})(3000)} = 0.08727 \times 10^3$$

$$c_8 = \frac{\pi d_8^4}{128\, \mu_8 \ell_8} = \frac{\pi (3^4)}{128\,(1.5\times10^{-6})(2000)} = 0.6627 \times 10^3$$

$$c_9 = \frac{\pi d_9^4}{128\, \mu_9 \ell_9} = \frac{\pi (2.5^4)}{128\,(1.5\times10^{-6})(6000)} = 0.1065 \times 10^3$$

$$c_{10} = \frac{\pi d_{10}^4}{128\, \mu_{10} \ell_{10}} = \frac{\pi (2.5^4)}{128\,(1.5\times10^{-6})(6000)} = 0.1065 \times 10^3$$

$$[K^{(e)}] = c_e \begin{array}{cc} i & j \\ \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] & \begin{array}{c} i \\ j \end{array} \end{array}$$

: Element characteristic matrix of element e

$(e = 1, 2, \ldots, 10)$

Assembled characteristic matrix:

$$
[\underset{\sim}{K}] = 10^{3} * \begin{array}{c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\left[\begin{array}{cccccccccc}
0.2651 & 0 & 0 & -0.2651 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0374 & -0.0374 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.0374 & \begin{smallmatrix}0.0374\\ 0.1473\\ 0.00327\end{smallmatrix} & -0.1473 & -0.00327 & 0 & 0 & 0 & 0 & 0 \\
-0.2651 & 0 & -0.1473 & \begin{smallmatrix}0.2651\\ 0.1473\\ 0.0436\end{smallmatrix} & 0 & 0 & 0 & -0.0436 & 0 & 0 \\
0 & 0 & -0.00327 & 0 & \begin{smallmatrix}0.00327\\ 0.08727\\ 0.6627\end{smallmatrix} & -0.6627 & -0.08727 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.6627 & 0.6627 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.08727 & 0 & \begin{smallmatrix}0.0799\\ 0.08727\\ 0.1065\end{smallmatrix} & -0.0799 & -0.1065 & 0 \\
0 & 0 & 0 & -0.0436 & 0 & 0 & -0.0799 & \begin{smallmatrix}0.0436\\ 0.0799\\ 0.1065\end{smallmatrix} & 0 & -0.1065 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.1065 & 0 & 0.1065 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1065 & 0 & 0.1065
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}
$$

where the nodes of different elements are assumed as indicated below:

Element	e	1	2	3	4	5	6	7	8	9	10
Node	i	1	3	4	4	8	3	5	5	7	8
Node	j	4	2	3	8	7	5	7	6	9	10

System equations can be expressed as

$$[\underset{\sim}{K}]\ \underset{\sim}{\vec{P}} = \underset{\sim}{\vec{Q}} \qquad (1)$$

where

$\vec{P}$ = vector of nodal pressures

$$= \{ p_1\ p_2\ p_3\ p_4\ p_5\ p_6\ p_7\ p_8\ p_9\ p_{10} \}^T$$

$$\equiv \{ 22\ \ 20\ \ p_3\ \ p_4\ \ p_5\ \ 18\ \ p_7\ \ p_8\ \ 14\ \ 17 \}^T$$

and

$\vec{Q}$ = vector of nodal external flow rates

$$= \{ Q_1\ Q_2\ Q_3\ Q_4\ Q_5\ Q_6\ Q_7\ Q_8\ Q_9\ Q_{10} \}^T$$

$$\equiv \{ Q_1\ Q_2\ 0\ 0\ 0\ Q_6\ 0\ 0\ Q_9\ Q_{10} \}^T$$

By using the known nodal pressures, Eqs. (1) can be expressed with $\vec{P} = \{ p_3\ p_4\ p_5\ p_7\ p_8 \}^T$ as the vector of unknowns as

$$\begin{bmatrix} 0.18797 & -0.1473 & -0.00327 & 0 & 0 \\ -0.1473 & 0.4560 & 0 & 0 & -0.0436 \\ -0.00327 & 0 & 0.75324 & -0.08727 & 0 \\ 0 & 0 & -0.08727 & 0.27367 & -0.0799 \\ 0 & -0.0436 & 0 & -0.0799 & 0.2300 \end{bmatrix} \begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_7 \\ p_8 \end{Bmatrix} = \begin{Bmatrix} 0.7480 \\ 0 \\ 11.9286 \\ 1.4910 \\ 1.8105 \end{Bmatrix} \qquad (2)$$

Solution of Eqs. (2) gives the pressures:

$$\{ p_3\ p_4\ p_5\ p_7\ p_8 \} \equiv \{ 7.1231\ \ 3.6206\ \ 17.6163\ \ 15.0954\ \ 13.8021 \}$$

1.33

Using the relation

$$\vec{I} = \left[K^{(e)}\right]\vec{V} = \frac{1}{R^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} V_i \\ V_j \end{bmatrix}$$

The characteristic matrices for each element are

$$\left[K^{(1)}\right] = \frac{1}{10}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ for nodes 1 and 2}$$

$$\left[K^{(2)}\right] = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ for nodes 3 and 4}$$

$$\left[K^{(3)}\right] = \frac{1}{6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ for nodes 2 and 3}$$

$$\left[K^{(4)}\right] = \frac{1}{8}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ for nodes 2 and 3}$$

Assembled into the global matrix

$$[K] = \begin{bmatrix} 0.1000 & -0.1000 & 0 & 0 \\ -0.1000 & 0.3917 & -0.2917 & 0 \\ 0 & -0.2917 & 0.5417 & -0.2500 \\ 0 & 0 & -0.2500 & 0.2500 \end{bmatrix}$$

We know that

$$\vec{V} = \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{Bmatrix} = \begin{Bmatrix} 40 \\ V_2 \\ V_3 \\ 0 \end{Bmatrix}$$

The system equations can then be expressed as

$$\begin{bmatrix} 0.1000 & 0 & 0 & 0 \\ 0 & 0.3917 & -0.2917 & 0 \\ 0 & -0.2917 & 0.5417 & 0 \\ 0 & 0 & 0 & 0.2500 \end{bmatrix}\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{Bmatrix}$$

This gives

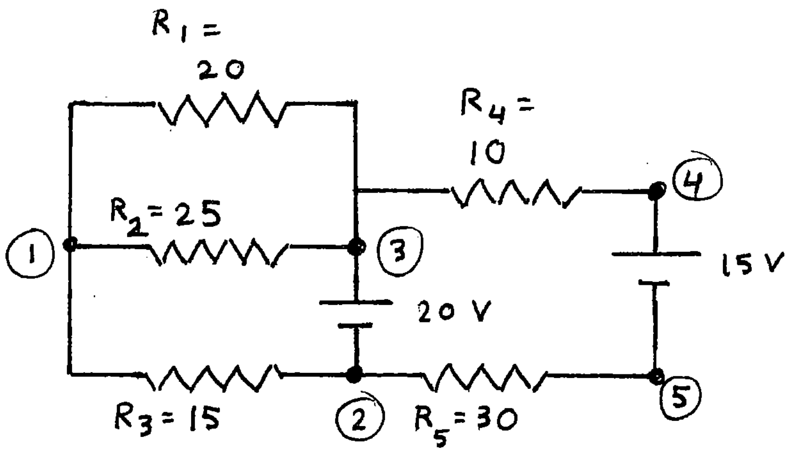
$$V_2 = 17 \text{ V}$$
$$V_3 = 9 \text{ V}$$

Therefore the currents are

$$\vec{I}=\begin{Bmatrix} I^{(1)} \\ I^{(2)} \\ I^{(3)} \\ I^{(4)} \end{Bmatrix}=\begin{Bmatrix} \dfrac{V_1-V_2}{R_1} \\ \dfrac{V_3-V_4}{R_2} \\ \dfrac{V_2-V_3}{R_3} \\ \dfrac{V_2-V_3}{R_4} \end{Bmatrix}=\begin{Bmatrix} 2.2951 \\ 2.2951 \\ 1.3115 \\ 0.9836 \end{Bmatrix}\text{A}$$

As a check, we can see that the total current remains constant $I^{(1)}=I^{(2)}=I^{(3)}+I^{(4)}$.

1.34



Element characteristic matrices :

$$\left[K^{(i)}\right] = \frac{1}{R_i}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad ; \ i = 1, 2, 3, 4, 5 \tag{1}$$

with $R_1 = 20$, $R_2 = 25$, $R_3 = 15$, $R_4 = 10$ and $R_5 = 30$.

Assembled characteristic matrix :

$$[\underset{\sim}{K}] = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} 0.05+0.04+0.0667 & -0.0667 & -0.05-0.04 & 0 & 0 \\ -0.0667 & 0.0667+0.0333 & 0 & 0 & -0.0333 \\ -0.05-0.04 & 0 & 0.05+0.04+0.1 & -0.1 & 0 \\ 0 & 0 & -0.1 & 0.1 & 0 \\ 0 & -0.0333 & 0 & 0 & 0.0333 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

system equations :

$$\begin{bmatrix} 0.1567 & -0.0667 & -0.09 & 0 & 0 \\ -0.0667 & 0.10 & 0 & 0 & -0.0333 \\ -0.09 & 0 & 0.19 & -0.10 & 0 \\ 0 & 0 & -0.10 & 0.10 & 0 \\ 0 & -0.0333 & 0 & 0 & 0.0333 \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2}$$

Since V_1 is the only unknown in Eq. (2), with $V_2 = 0$, $V_3 = 20$, $V_4 = 15$ and $V_5 = 0$, we express the first equation of Eq. (2) as

$$0.1567\, V_1 - 0.0667\, V_2 - 0.09\, V_3 = 0$$

or

$$0.1567\, V_1 = 0.0667\, V_2 + 0.09\, V_3 = 0.09(20) = 1.8$$

or

$$V_1 = 11.4869 \text{ V}$$

The currents in elements can be computed as

$$I_1 = \frac{V_3 - V_1}{R_1} = \frac{20 - 11.4869}{20} = 0.4256 \text{ Ampere}$$

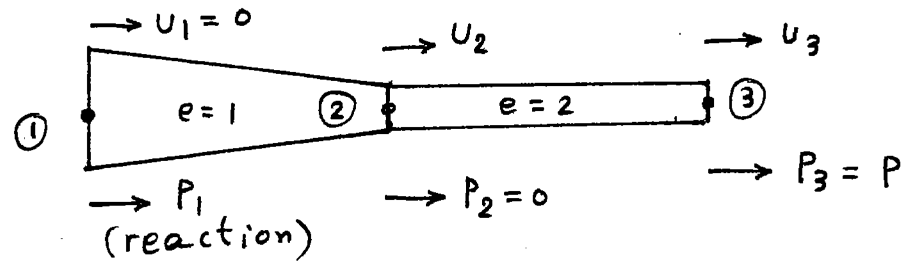
$$I_2 = \frac{V_3 - V_1}{R_2} = \frac{20 - 11.4869}{25} = 0.3405 \text{ Ampere}$$

$$I_3 = \frac{V_3 - V_1}{R_3} = \frac{20 - 11.4869}{15} = 0.5675 \text{ Ampere}$$

$$I_4 = \frac{V_3 - V_4}{R_4} = \frac{20 - 15}{10} = 0.5 \text{ Ampere}$$

$$I_5 = \frac{V_2 - V_5}{R_5} = \frac{0 - 0}{30} = 0 \text{ Ampere}$$

1.35



Element stiffness matrices:

From Solution of Problem 1.10, element stiffness matrix of a linearly tapered bar is given by

$$[K^{(e)}] = \left(\frac{A_1^{(e)} + A_2^{(e)}}{2}\right) \frac{E^{(e)}}{l^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1}$$

with

$$A^{(e)}(x) = A_1^{(e)} + \frac{A_2^{(e)} - A_1^{(e)}}{l^{(e)}}\, x \tag{2}$$

Here $A_1^{(1)} = 4 \text{ in}^2$ and $A_2^{(1)} = 2 \text{ in}^2$ and $\frac{A_1^{(1)} + A_2^{(1)}}{2} = 3 \text{ in}^2$.

$$[K^{(1)}] = \frac{3\,(30 \times 10^6)}{40} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2.25 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ lb/in}$$

$$[K^{(2)}] = \frac{2\,(30 \times 10^6)}{30} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2.0 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ lb/in}$$

Assembled stiffness matrix of the system:

$$[\underset{\sim}{K}] = 10^6 \begin{array}{c} \begin{array}{ccc} U_1 & U_2 & U_3 \end{array} \\ \begin{bmatrix} 2.25 & -2.25 & 0 \\ -2.25 & 2.25 + 2.0 & -2.0 \\ 0 & -2.0 & 2.0 \end{bmatrix} \end{array} \begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \end{array} \tag{3}$$

system equilibrium equations before applying b.c.s:

$$10^6 \begin{bmatrix} 2.25 & -2.25 & 0 \\ -2.25 & 4.25 & -2.0 \\ 0 & -2.0 & 2.0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 100 \end{Bmatrix} \tag{4}$$

system equilibrium equations after applying the b.c., $U_1 = 0$ (by deleting the row and column corresponding to U_1 in Eq. (4)):

$$10^6 \begin{bmatrix} 4.25 & -2.0 \\ -2.0 & 2.0 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 100 \end{Bmatrix} \qquad (5)$$

solution of Eq. (5):

$$\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = 10^{-4} \begin{Bmatrix} 0.4444 \\ 0.9444 \end{Bmatrix} \text{ in} \qquad (6)$$

Average stress in element 1:

$$\sigma_{av}^{(1)} = E^{(1)} \epsilon_{av}^{(1)} = E^{(1)} \left(\frac{U_2 - U_1}{l^{(1)}} \right)$$

$$= 30 \times 10^6 \left(\frac{0.4444 - 0}{40} \right) 10^{-4}$$

$$= 33.33 \ lb/in^2 \qquad (7)$$

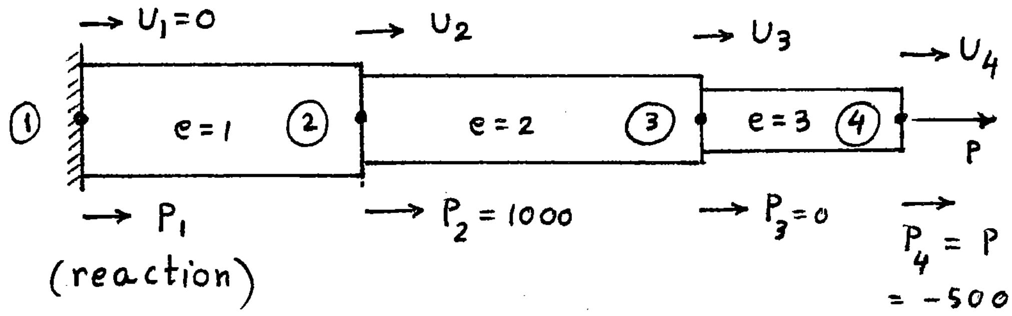
Stress in element 2:

$$\sigma^{(2)} = E^{(2)} \epsilon^{(2)} = E^{(2)} \left(\frac{U_3 - U_2}{l^{(2)}} \right)$$

$$= 30 \times 10^6 \left(\frac{0.9444 - 0.4444}{30} \right) 10^{-4}$$

$$= 50.00 \ lb/in^2 \qquad (8)$$

1.36



Element stiffness matrices:

$$[K^{(1)}] = \frac{A_1 E_1}{\ell_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(15\times10^{-4})(207\times10^{9})}{0.6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 5175.0\times10^{5} \begin{array}{cc} U_1 & U_2 \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{array}{c} U_1 \\ U_2 \end{array} \ N/m \qquad (1)$$

$$[K^{(2)}] = \frac{A_2 E_2}{\ell_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(10\times10^{-4})(71\times10^{9})}{0.8}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 887.5\times10^{5} \begin{array}{cc} U_2 & U_3 \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{array}{c} U_2 \\ U_3 \end{array} \ N/m \qquad (2)$$

$$[K^{(3)}] = \frac{A_3 E_3}{\ell_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(5\times10^{-4})(119\times10^{9})}{0.4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 1487.5\times10^{5} \begin{array}{cc} U_3 & U_4 \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{array}{c} U_3 \\ U_4 \end{array} \ N/m \qquad (3)$$

Assembled stiffness matrix:

$$[\underset{\sim}{K}] = 10^{5}\begin{bmatrix} 5175.0 & -5175.0 & 0 & 0 \\ -5175.0 & (5175.0+887.5) & -887.5 & 0 \\ 0 & -887.5 & (887.5+1487.5) & -1487.5 \\ 0 & 0 & -1487.5 & 1487.5 \end{bmatrix}\begin{array}{c} U_1 \\ U_2 \\ U_3 \\ U_4 \end{array}$$

$$\begin{array}{cccc} U_1 & U_2 & U_3 & U_4 \end{array}$$

System equilibrium equations before applying boundary conditions:

$$10^5 \begin{bmatrix} 5175.0 & -5175.0 & 0 & 0 \\ -5175.0 & 6062.5 & -887.5 & 0 \\ 0 & -887.5 & 2375.0 & -1487.5 \\ 0 & 0 & -1487.5 & 1487.5 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 1000 \\ 0 \\ -500 \end{Bmatrix} \quad (4)$$

System equations after applying the boundary condition, $U_1 = 0$:

$$10^5 \begin{bmatrix} 6062.5 & -887.5 & 0 \\ -887.5 & 2375.0 & -1487.5 \\ 0 & -1487.5 & 1487.5 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 0 \\ -500 \end{Bmatrix} \quad (5)$$

Solution of Eq. (5):

$U_2 = 0.0966 \times 10^{-5}$ m, $U_3 = -0.4668 \times 10^{-5}$ m, $U_4 = -0.8029 \times 10^{-5}$ m

Stress in element 1:

$$\sigma_1 = E_1 \epsilon_1 = E_1 \left(\frac{U_2 - U_1}{l_1} \right) = 207 \times 10^9 \left(\frac{0.0966 - 0}{0.6} \right) 10^{-5} = 33.327 \times 10^4 \text{ N/m}^2$$

Stress in element 2:

$$\sigma_2 = E_2 \epsilon_2 = E_2 \left(\frac{U_3 - U_2}{l_2} \right) = 71 \times 10^9 \left(\frac{-0.4668 + 0.0966}{0.8} \right) 10^{-5}$$

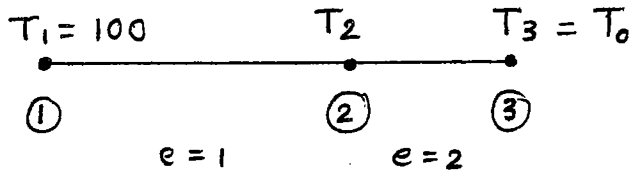
$$= -32.8552 \times 10^4 \text{ N/m}^2$$

Stress in element 3:

$$\sigma_3 = E_3 \epsilon_3 = E_3 \left(\frac{U_4 - U_3}{l_3} \right) = 119 \times 10^9 \left(\frac{-0.8029 + 0.4668}{0.4} \right) 10^{-5}$$

$$= -99.9897 \times 10^4 \text{ N/m}^2$$

1.37



Element characteristic matrices:

$$[K^{(i)}] = k_e^{(i)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \; ; \; i = 1, 2$$

$$k_1^{(e)} = \frac{k_e'}{t_e} = \frac{0.4}{15} = 0.02667 \; ; \; k_2^{(e)} = \frac{0.2}{5} = 0.04$$

$$\begin{array}{ccc} & 1 & 2 & \\ [K^{(1)}] = 0.02667 & \left[\begin{array}{c} 1 \\ -1 \end{array}\right. & \left.\begin{array}{c} -1 \\ 1 \end{array}\right] & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$$\begin{array}{cccc} & 2 & 3 & \\ [K^{(2)}] = 0.04 & \left[\begin{array}{c} 1 \\ -1 \end{array}\right. & \left.\begin{array}{c} -1 \\ 1 \end{array}\right] & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

Assembled characteristic matrix:

$$[\underset{\sim}{K}] = \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{cccc} 1 & 2 & 3 & \\ \left[\begin{array}{c} 0.02667 \\ -0.02667 \\ 0 \end{array}\right. & \begin{array}{c} -0.02667 \\ (0.02667 + 0.04) \\ -0.04 \end{array} & \left.\begin{array}{c} 0 \\ -0.04 \\ 0.04 \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

System equations before applying boundary condition:

$$\begin{bmatrix} 0.02667 & -0.02667 & 0 \\ -0.02667 & 0.06667 & -0.04 \\ 0 & -0.04 & 0.04 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} F_1 = 0 \\ F_2 = 0 \\ F_3 = 0 \end{Bmatrix} \quad (1)$$

Since T_2 is the only unknown, the middle equation of Eq. (1) can be rewritten as

$$-0.02667 \; T_1 + 0.06667 \; T_2 - 0.04 \; T_3 = 0$$

or

$$0.06667 \; T_2 = 0.02667 \underset{100}{\underset{\shortparallel}{T_1}} + 0.04 \underset{70}{\underset{\shortparallel}{T_3}} = 5.467$$

or $$T_2 = 5.467 / 0.06667 = 82^\circ$$

1.38

$$T_1 = T_{high} = 200°C$$
$$T_5 = T_{low} = 50°C$$
$$\vec{F} = [K^{(e)}]\vec{T} = \frac{k_e}{t_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\vec{T}$$

For each element

$$[K^{(1)}] = \frac{0.4}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$[K^{(2)}] = \frac{0.3}{10}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$[K^{(3)}] = \frac{0.2}{15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$[K^{(4)}] = \frac{0.1}{8}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assembled into the global

$$[K] = \begin{bmatrix} 0.0800 & -0.0800 & 0 & 0 & 0 \\ -0.0800 & 0.1100 & -0.0300 & 0 & 0 \\ 0 & -0.0300 & 0.0433 & -0.0133 & 0 \\ 0 & 0 & -0.0133 & 0.0258 & -0.0125 \\ 0 & 0 & 0 & -0.0125 & 0.0125 \end{bmatrix}$$

We know that

$$\vec{T} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 200 \\ T_2 \\ T_3 \\ T_4 \\ 50 \end{Bmatrix}$$

Thus the system of equations can be written as

$$\begin{bmatrix} 0.0800 & 0 & 0 & 0 & 0 \\ 0 & 0.1100 & -0.0300 & 0 & 0 \\ 0 & -0.0300 & 0.0433 & -0.0133 & 0 \\ 0 & 0 & -0.0133 & 0.0258 & 0 \\ 0 & 0 & 0 & 0 & 0.0125 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 16 \\ 0 \\ 0 \\ 0 \\ 0.625 \end{Bmatrix}$$

Here, $\vec{F}$ represents the vector of heat fluxes applied at the nodes. We make the assumption that there are no heat fluxes (i.e. no heat sources or sinks) within the wall so that F_2, F_3 and F_4 are zero. Therefore the temperatures are

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 200 \\ 190.6639 \\ 165.7677 \\ 109.7510 \\ 50 \end{Bmatrix} °\mathrm{C}$$
