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# 1

## Vectors, Tensors, and Equations of Elasticity

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**1.1** Prove the following properties of  $\delta_{ij}$  and  $\varepsilon_{ijk}$  (assume  $i, j = 1, 2, 3$  when they are dummy indices):

(a)  $F_{ij}\delta_{jk} = F_{ik}$

(b)  $\delta_{ij}\delta_{ij} = \delta_{ii} = 3$

(c)  $\varepsilon_{ijk}\varepsilon_{ijk} = 6$

(d)  $\varepsilon_{ijk}F_{ij} = 0$  whenever  $F_{ij} = F_{ji}$  (symmetric)

*Solution:*

**1.1(a)** Expanding the expression

$$F_{ij}\delta_{jk} = F_{i1}\delta_{1k} + F_{i2}\delta_{2k} + F_{i3}\delta_{3k}$$

Of the three terms on the right hand side, only one is nonzero. It is equal to  $F_{i1}$  if  $k = 1$ ,  $F_{i2}$  if  $k = 2$ , or  $F_{i3}$  if  $k = 3$ . Thus, it is simply equal to  $F_{ik}$ .

**1.1(b)** By actual expansion, we have

$$\begin{aligned}\delta_{ij}\delta_{ij} &= \delta_{i1}\delta_{i1} + \delta_{i2}\delta_{i2} + \delta_{i3}\delta_{i3} \\ &= (\delta_{11}\delta_{11} + 0 + 0) + (0 + \delta_{22}\delta_{22} + 0) + (0 + 0 + \delta_{33}\delta_{33}) \\ &= 3\end{aligned}$$

and

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$$

Alternatively, using  $F_{ij} = \delta_{ij}$  in Problem 1.1a, we have  $\delta_{ij}\delta_{jk} = \delta_{ik}$ , where  $i$  and  $k$  are free indices that can any value. In particular, for  $i = k$ , we have the required result.

**1.1(c)** Using the  $\varepsilon$ - $\delta$  identity and the result of Problem 1.1(b), we obtain

$$\varepsilon_{ijk}\varepsilon_{ijk} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ij} = 9 - 3 = 6$$

1.1(d) We have

$$\begin{aligned} F_{ij}\varepsilon_{ijk} &= -F_{ij}\varepsilon_{jik} \quad (\text{interchanged } i \text{ and } j) \\ &= -F_{ji}\varepsilon_{ijk} \quad (\text{renamed } i \text{ as } j \text{ and } j \text{ as } i) \end{aligned}$$

Since  $F_{ji} = F_{ij}$ , we have

$$\begin{aligned} 0 &= (F_{ij} + F_{ji})\varepsilon_{ijk} \\ &= 2F_{ij}\varepsilon_{ijk} \end{aligned}$$

The converse also holds, i.e., if  $F_{ij}\varepsilon_{ijk} = 0$ , then  $F_{ij} = F_{ji}$ . We have

$$\begin{aligned} 0 &= F_{ij}\varepsilon_{ijk} \\ &= \frac{1}{2}(F_{ij}\varepsilon_{ijk} + F_{ij}\varepsilon_{ijk}) \\ &= \frac{1}{2}(F_{ij}\varepsilon_{ijk} - F_{ij}\varepsilon_{jik}) \quad (\text{interchanged } i \text{ and } j) \\ &= \frac{1}{2}(F_{ij}\varepsilon_{ijk} - F_{ji}\varepsilon_{ijk}) \quad (\text{renamed } i \text{ as } j \text{ and } j \text{ as } i) \\ &= \frac{1}{2}(F_{ij} - F_{ji})\varepsilon_{ijk} \end{aligned}$$

from which it follows that  $F_{ji} = F_{ij}$ .

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♠ **New Problem 1.1:** Show that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

*Solution:* Write the position vector in cartesian component form using the index notation

$$\mathbf{r} = x_j \hat{\mathbf{e}}_j \tag{1}$$

Then the square of the magnitude of the position vector is

$$\begin{aligned} r^2 &= \mathbf{r} \cdot \mathbf{r} = (x_i \hat{\mathbf{e}}_i) \cdot (x_j \hat{\mathbf{e}}_j) = x_i x_j \delta_{ij} \\ &= x_i x_i = x_k x_k \end{aligned} \tag{2}$$

Its derivative of  $r$  with respect to  $x_i$  can be obtained from

$$\begin{aligned} \frac{\partial r^2}{\partial x_i} &= \frac{\partial}{\partial x_i} (x_k x_k) \\ &= \frac{\partial x_k}{\partial x_i} x_k + x_k \frac{\partial x_k}{\partial x_i} \\ &= 2 \frac{\partial x_k}{\partial x_i} x_k = 2\delta_{ik} x_k = 2x_i \end{aligned}$$

Hence

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \tag{3}$$


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**1.2** Let  $\mathbf{r}$  denote a position vector. Show that:

- (a)  $\text{grad}(r^n) = nr^{n-2}\mathbf{r}$
- (b)  $\nabla^2(r^n) = n(n+1)r^{n-2}$
- (c)  $\text{div}(\mathbf{r}) = 3$
- (d)  $\text{curl}(\mathbf{r}f(r)) = \mathbf{0}$ , where  $f(r)$  is an arbitrary continuous function of  $r$  with continuous first derivatives

*Solution:*

**1.2(a)** We have

$$\nabla(r^n) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} (r^n) = nr^{n-1} \hat{\mathbf{e}}_i \frac{\partial r}{\partial x_i} = nr^{n-2} x_i \hat{\mathbf{e}}_i = nr^{n-2} \mathbf{r}$$

where the result from Eq. (3) of Problem 1.1 is used in arriving at the last step.

**1.2(b)** From the result of the above exercise, we have

$$\begin{aligned} \nabla^2(r^n) &= (\nabla \cdot \nabla)(r^n) = \nabla \cdot [\nabla(r^n)] \\ &= \left( \hat{\mathbf{e}}_j \frac{\partial}{\partial x_j} \right) \cdot (nr^{n-2} x_i \hat{\mathbf{e}}_i) = n(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_i) \frac{\partial}{\partial x_j} (r^{n-2} x_i) \\ &= n\delta_{ij} \left[ (n-2)r^{n-3} \frac{x_j}{r} x_i + r^{n-2} \delta_{ij} \right] \\ &= n \left[ (n-2)r^{n-2} + 3r^{n-2} \right] = n(n+1)r^{n-2} \end{aligned}$$

**1.2(c)** Using Eq. (3) of Problem 1.1(b), we obtain

$$\nabla \cdot \mathbf{r} = \left( \hat{\mathbf{e}}_j \frac{\partial}{\partial x_j} \right) \cdot (x_i \hat{\mathbf{e}}_i) = (\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_i) \frac{\partial x_i}{\partial x_j} = \delta_{ij} \delta_{ij} = 3$$

**1.2(d)** We obtain

$$\begin{aligned} \text{curl}(\mathbf{r}f) &= \nabla \times [f\mathbf{r}] \\ &= \left( \hat{\mathbf{e}}_j \frac{\partial}{\partial x_j} \right) \times (f(r)x_i \hat{\mathbf{e}}_i) = (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_i) \frac{\partial}{\partial x_j} [f(r)x_i] \\ &= \varepsilon_{jik} \hat{\mathbf{e}}_k \left[ f'(r) \frac{\partial r}{\partial x_j} x_i + f(r) \delta_{ij} \right] \\ &= \varepsilon_{jik} \hat{\mathbf{e}}_k \left[ \frac{f'(r)}{r} x_j x_i + f(r) \delta_{ij} \right] \end{aligned}$$

The two terms in the square brackets,  $x_i x_j$  and  $\delta_{ij}$  are symmetric, hence, by Problem 1.1(d) the expression in the last line is zero.

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♠ **New Problem 1.2:** Let  $[A]$  and  $[B]$  be  $m \times n$  and  $n \times p$  matrices, respectively. Show that

$$([A][B])^T = [B]^T[A]^T \quad (1)$$

*Solution:* By definition of the product of two matrices, we have  $[A][B] = [C]$  with

$$c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$$

Then the transpose of  $[C]$  has the coefficients

$$\begin{aligned} c_{ji} &= \sum_{k=1}^n a_{jk}b_{ki} = \sum_{k=1}^n b_{ki}a_{jk} \\ &= \sum_{k=1}^n (b_{ik})^T (a_{kj})^T \end{aligned}$$

which implies the result in Eq. (1).

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**1.3** If  $[B]$  is a symmetric  $n \times n$  matrix and  $[C]$  is any  $n \times n$  matrix, show that  $[C]^T[B][C]$  is symmetric.

*Solution:* Let  $[A] = [B][C]$ . Using Eq. (1) of New Problem 1.2, we have

$$([C]^T[A])^T = [A]^T[C] = [C]^T[B][C]$$

where we have also used the identity

$$([C]^T)^T = [C]$$


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♠ **New Problem 1.3:** Show that the dot and cross can be interchanged without changing the value in the *scalar triple product*

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} \quad (1)$$

*Solution:* We have

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} \times \mathbf{C} &= A_i \hat{\mathbf{e}}_i \cdot B_j C_k \varepsilon_{jkm} \hat{\mathbf{e}}_m = A_i B_j C_k \varepsilon_{jkm} \delta_{im} \\ &= A_i B_j C_k \varepsilon_{jki} = A_i B_j C_k \varepsilon_{ijk} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} \end{aligned}$$

Since  $i$ ,  $j$ , and  $k$  can be permuted in a cyclic order, it also follows that

$$A_i B_j C_k \varepsilon_{ijk} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A}$$

and  $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{B} \times \mathbf{C} \cdot \mathbf{A}$ .

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1.4 Establish the  $\varepsilon$ - $\delta$  identity of Eq. (1.2.15).

*Solution:* The  $\varepsilon$ - $\delta$  identity follows directly from the vector identity

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \quad (1)$$

by letting

$$\mathbf{A} = \hat{\mathbf{e}}_i, \quad \mathbf{B} = \hat{\mathbf{e}}_j, \quad \mathbf{C} = \hat{\mathbf{e}}_m, \quad \mathbf{D} = \hat{\mathbf{e}}_n$$

We obtain

$$(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot (\hat{\mathbf{e}}_m \times \hat{\mathbf{e}}_n) = (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_m)(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_n) - (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_n)(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_m)$$

$$\varepsilon_{ijk} \hat{\mathbf{e}}_k \cdot \varepsilon_{mnp} \hat{\mathbf{e}}_p = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}$$

or

$$\varepsilon_{ijk} \varepsilon_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}$$

which was to be proved. Note that  $\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$ .

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1.5 Prove that the determinant of a  $3 \times 3$  matrix  $[C]$  can be expressed in the form

$$|C| = \varepsilon_{ijk} c_{1i} c_{2j} c_{3k} \quad (a)$$

and, thus, prove

$$|C| = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{rst} c_{ir} c_{js} c_{kt} \quad (b)$$

where  $c_{ij}$  is the element occupying the  $i$ th row and the  $j$ th column of  $[C]$ .

*Solution:* First we note the definition of the cross product of two vectors

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \quad (3)$$

and the “scalar triple product” of vectors

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \quad (4)$$

Now let

$$\mathbf{A} = c_{1i} \hat{\mathbf{e}}_i \equiv \mathbf{C}_1, \quad \mathbf{B} = c_{2j} \hat{\mathbf{e}}_j \equiv \mathbf{C}_2, \quad \mathbf{C} = c_{3k} \hat{\mathbf{e}}_k \equiv \mathbf{C}_3$$

in Eq. (3). We obtain

$$\begin{aligned} \mathbf{C}_1 \cdot (\mathbf{C}_2 \times \mathbf{C}_3) &= c_{1i} \hat{\mathbf{e}}_i \cdot (c_{2j} \hat{\mathbf{e}}_j \times c_{3k} \hat{\mathbf{e}}_k) \\ &= \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} \equiv |C| \end{aligned}$$

or

$$\begin{aligned} |C| &= c_{1i}\hat{\mathbf{e}}_i \cdot (c_{2j}\hat{\mathbf{e}}_j \times c_{3k}\hat{\mathbf{e}}_k) \\ &= c_{1i}c_{2j}c_{3k}\varepsilon_{ijk} \end{aligned}$$

which is the same as Eq. (1). Next consider the product  $\mathbf{C}_r \cdot (\mathbf{C}_s \times \mathbf{C}_t)$ :

$$\mathbf{C}_r \cdot (\mathbf{C}_s \times \mathbf{C}_t) = c_{ri}c_{sj}c_{tk}\varepsilon_{ijk}$$

Multiplying both sides with  $\varepsilon_{rst}$  and expanding, we arrive at

$$\begin{aligned} c_{ri}c_{sj}c_{tk}\varepsilon_{rst}\varepsilon_{ijk} &= \varepsilon_{rst}[\mathbf{C}_r \cdot (\mathbf{C}_s \times \mathbf{C}_t)] \\ &= \varepsilon_{1st}[\mathbf{C}_1 \cdot (\mathbf{C}_s \times \mathbf{C}_t)] + \varepsilon_{2st}[\mathbf{C}_2 \cdot (\mathbf{C}_s \times \mathbf{C}_t)] \\ &\quad + \varepsilon_{3st}[\mathbf{C}_3 \cdot (\mathbf{C}_s \times \mathbf{C}_t)] \\ &= \mathbf{C}_1 \cdot (\mathbf{C}_2 \times \mathbf{C}_3) - \mathbf{C}_1 \cdot (\mathbf{C}_3 \times \mathbf{C}_2) \\ &\quad + \mathbf{C}_2 \cdot (\mathbf{C}_3 \times \mathbf{C}_1) - \mathbf{C}_2 \cdot (\mathbf{C}_1 \times \mathbf{C}_3) \\ &\quad + \mathbf{C}_3 \cdot (\mathbf{C}_1 \times \mathbf{C}_2) - \mathbf{C}_3 \cdot (\mathbf{C}_2 \times \mathbf{C}_1) \\ &= 6[\mathbf{C}_1 \cdot (\mathbf{C}_2 \times \mathbf{C}_3)] = 6|C| \end{aligned}$$

where we have used the identity in Eq. (1) of New Problem 1.3.

**1.6** Using Cramer's rule determine the solution to the following equations:

(a)

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 2x_2 - x_3 &= 2 \\ -x_2 + 2x_3 &= 2 \end{aligned}$$

(b)

$$\frac{2b}{h^3} \begin{bmatrix} 12 & 0 & 3h \\ 0 & 4h^2 & h^2 \\ 3h & h^2 & 2h^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{f_0 h}{12} \begin{Bmatrix} 12 \\ 0 \\ h \end{Bmatrix}$$

where  $b$ ,  $f_0$ , and  $h$  are constants

*Solution:*

**1.6(a)** The matrix form of the equations is

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix}$$

Using Cramer's rule we obtain

$$\begin{aligned} x_1 &= \frac{1}{|A|} \begin{vmatrix} 1 & -1 & 0 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{vmatrix} = \frac{1}{|A|} [(4 - 1) + (4 + 2) - 0] = \frac{9}{|A|} \\ x_2 &= \frac{1}{|A|} \begin{vmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & 2 & 2 \end{vmatrix} = \frac{1}{|A|} [2(4 + 2) - (-2 - 0) - 0] = \frac{14}{|A|} \\ x_3 &= \frac{1}{|A|} \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & 2 \\ 0 & -1 & 2 \end{vmatrix} = \frac{1}{|A|} [2(4 + 2) + (-2 + 1) + 0] = \frac{11}{|A|} \end{aligned}$$

where the determinant  $|A|$  of the coefficient matrix is

$$|A| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2(4 - 1) + (-2 - 0) - 0 = 4$$

Hence,  $x_1 = 9/4$ ,  $x_2 = 14/4$ , and  $x_3 = 11/4$ .

**1.6(b)** We have

$$\begin{bmatrix} 12 & 0 & 3h \\ 0 & 4h^2 & h^2 \\ 3h & h^2 & 2h^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{f_0 h^4}{24b} \begin{Bmatrix} 12 \\ 0 \\ h \end{Bmatrix}$$

The determinant of the coefficient matrix is  $|A| = 12(8h^4 - h^4) + 3h(-12h^3) = 48h^4$ . Using Cramer's rule we obtain

$$x_1 = \frac{\alpha}{|A|} \begin{vmatrix} 12 & 0 & 3h \\ 0 & 4h^2 & h^2 \\ h & h^2 & 2h^2 \end{vmatrix} = \frac{\alpha}{|A|} [12(8h^4 - h^4) + h(-12h^3)] = \frac{72\alpha}{|A|}$$

$$x_2 = \frac{\alpha}{|A|} \begin{vmatrix} 12 & 12 & 3h \\ 0 & 0 & h^2 \\ 3h & h & 2h^2 \end{vmatrix} = \frac{\alpha}{|A|} [12(-h^3) + 3h(12h^2)] = \frac{24\alpha}{|A|}$$

$$x_3 = \frac{1}{|A|} \begin{vmatrix} 12 & 0 & 12 \\ 0 & 4h^2 & 0 \\ 3h & h^2 & h \end{vmatrix} = \frac{\alpha}{|A|} [12(4h^3) + 3h(-48h^2)] = -\frac{96\alpha}{|A|}$$

Hence,  $x_1 = 3\alpha/2$ ,  $x_2 = \alpha/(2h)$ , and  $x_3 = -2\alpha/h$ , where  $\alpha = (f_0 h^4/24b)$ .

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**1.7** Let  $[C]$  be a  $3 \times 3$  matrix,  $[I]$  be a  $3 \times 3$  identity matrix, and  $\lambda$  be a scalar. Show that

$$\det[C - \lambda I] = \lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3$$

where

$$I_1 = c_{ii}, \quad I_2 = \frac{1}{2}(c_{ii}c_{jj} - c_{ij}c_{ji}), \quad I_3 = |C|$$

*Solution:* Using the result of Problem 1.5(b) and the  $\varepsilon$ - $\delta$  identity, we obtain

$$\begin{aligned} |C - \lambda I| &= \frac{1}{6} \varepsilon_{ijk} \varepsilon_{rst} (c_{ir} - \lambda \delta_{ir})(c_{js} - \lambda \delta_{js})(c_{kt} - \lambda \delta_{kt}) \\ &= \frac{1}{6} \varepsilon_{ijk} \varepsilon_{rst} \left[ -\lambda^3 \delta_{ir} \delta_{js} \delta_{kt} + \lambda^2 (c_{ir} \delta_{js} \delta_{kt} + c_{kt} \delta_{ir} \delta_{js} + c_{js} \delta_{ir} \delta_{kt}) \right. \\ &\quad \left. - \lambda (c_{ir} c_{js} \delta_{kt} + c_{ir} \delta_{js} c_{kt} + \delta_{ir} c_{js} c_{kt}) + c_{ir} c_{js} c_{kt} \right] \\ &= -\lambda^3 + \frac{\lambda^2}{6} (c_{ir} \varepsilon_{ijk} \varepsilon_{rjk} + c_{kt} \varepsilon_{ijk} \varepsilon_{ijt} + \varepsilon_{ijk} \varepsilon_{isk} c_{js}) \\ &\quad + \frac{\lambda}{6} (\varepsilon_{ijk} \varepsilon_{rsk} c_{ir} c_{js} + \varepsilon_{ijk} \varepsilon_{rjt} c_{ir} c_{kt} + \varepsilon_{ijk} \varepsilon_{ist} c_{js} c_{kt}) \\ &\quad + \frac{1}{6} \varepsilon_{ijk} \varepsilon_{rst} c_{ir} c_{js} c_{kt} \\ &= -\lambda^3 + c_{ii} \lambda^2 + \frac{\lambda}{2} (c_{ii} c_{jj} - c_{ij} c_{ji}) + |C| \end{aligned}$$


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**1.8** If we identify a second-order tensor  $\mathbf{A}$  associated with the direction cosines  $a_{ij} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$  [see Eq. (1.2.57)]

$$\mathbf{A} = a_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j$$

show that (a)  $\mathbf{A} \cdot \mathbf{A} = \mathbf{A}$ , (b)  $\mathbf{A} \cdot \mathbf{A}^T = \mathbf{I}$ , and (c)  $\mathbf{A} : \mathbf{A} = 3$ .

*Solution:*

**1.8(a)** We have

$$\mathbf{A} \cdot \mathbf{A} = a_{ij} a_{kp} a_{kj} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_p = a_{ip} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_p = \mathbf{A}$$

where we have used the identity in Eq. (1.2.61).

**1.8(b)** We have

$$\mathbf{A} \cdot \mathbf{A}^T = a_{ij} a_{kp} \delta_{jp} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_k = a_{ij} a_{kj} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_k = \delta_{ip} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_k = \mathbf{I}$$

where we have used the identity in Eq. (1.2.61).

**1.8(c)** We have

$$\begin{aligned} \mathbf{A} : \mathbf{A} &= a_{ij} a_{mn} \left( \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_n \right) \left( \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_m \right) \\ &= a_{ij} a_{mn} a_{in} a_{mj} = \delta_{jn} \delta_{nj} = \delta_{jj} = 3 \end{aligned}$$

where we have used the identity in Eq. (1.2.61) repeatedly.

**1.9** Use the definition  $\nabla^2 = \nabla \cdot \nabla$  to show that the Laplacian operator in the cylindrical coordinate system is given by

$$\nabla^2 = \frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r} \frac{\partial^2}{\partial \theta^2} + r \frac{\partial^2}{\partial z^2} \right]$$

*Solution:* Using the definition (1.2.30) of  $\nabla$  and the derivatives of the basis vectors (1.2.29)

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}, \quad \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta, \quad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r$$

we obtain

$$\begin{aligned} \nabla \cdot \nabla &= \left( \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) \cdot \left( \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) \\ &= \hat{\mathbf{e}}_r \cdot \frac{\partial}{\partial r} \left( \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) \\ &\quad + \frac{1}{r} \hat{\mathbf{e}}_\theta \cdot \frac{\partial}{\partial \theta} \left( \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) \\ &\quad + \hat{\mathbf{e}}_z \cdot \frac{\partial}{\partial z} \left( \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \hat{\mathbf{e}}_\theta \cdot \left( \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r} \frac{\partial^2}{\partial \theta^2} \right) + \frac{\partial^2}{\partial z^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \end{aligned}$$


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**1.10** Show that the gradient of a vector  $\mathbf{u}$  in the cylindrical coordinate system is given by

$$\begin{aligned}\nabla \mathbf{u} &= \frac{\partial u_r}{\partial r} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_r + \frac{1}{r} \left( u_r + \frac{\partial u_\theta}{\partial \theta} \right) \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial z} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_z \\ &\quad + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_z + \frac{\partial u_\theta}{\partial z} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_\theta + \frac{\partial u_r}{\partial z} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_r + \frac{\partial u_z}{\partial r} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_z \\ &\quad + \frac{\partial u_\theta}{\partial r} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_\theta + \frac{1}{r} \left( \frac{\partial u_r}{\partial \theta} - u_\theta \right) \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r\end{aligned}$$

*Solution:* We have

$$\begin{aligned}\nabla \mathbf{u} &= \left( \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) (u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \mathbf{e}_z) \\ &= \hat{\mathbf{e}}_r \frac{\partial}{\partial r} (u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \mathbf{e}_z) \\ &\quad + \frac{\hat{\mathbf{e}}_\theta}{r} \frac{\partial}{\partial \theta} (u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \mathbf{e}_z) \\ &\quad + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} (u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \mathbf{e}_z) \\ &= \hat{\mathbf{e}}_r \left( \frac{\partial u_r}{\partial r} \hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial r} \hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial r} \mathbf{e}_z \right) \\ &\quad + \frac{\hat{\mathbf{e}}_\theta}{r} \left( \frac{\partial u_r}{\partial \theta} \hat{\mathbf{e}}_r + u_r \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} + \frac{\partial u_\theta}{\partial \theta} \hat{\mathbf{e}}_\theta + u_\theta \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} + \frac{\partial u_z}{\partial \theta} \hat{\mathbf{e}}_z \right) \\ &\quad + \hat{\mathbf{e}}_z \left( \frac{\partial u_r}{\partial z} \hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial z} \hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial z} \mathbf{e}_z \right) \\ &= \hat{\mathbf{e}}_r \left( \frac{\partial u_r}{\partial r} \hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial r} \hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial r} \mathbf{e}_z \right) \\ &\quad + \frac{\hat{\mathbf{e}}_\theta}{r} \left( \frac{\partial u_r}{\partial \theta} \hat{\mathbf{e}}_r + u_r \hat{\mathbf{e}}_\theta + \frac{\partial u_\theta}{\partial \theta} \hat{\mathbf{e}}_\theta - u_\theta \hat{\mathbf{e}}_r + \frac{\partial u_z}{\partial \theta} \hat{\mathbf{e}}_z \right) \\ &\quad + \hat{\mathbf{e}}_z \left( \frac{\partial u_r}{\partial z} \hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial z} \hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial z} \mathbf{e}_z \right)\end{aligned}$$

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**1.11** Show that the curl of a vector  $\mathbf{u}$  in the cylindrical coordinate system is given by

$$\begin{aligned}\nabla \times \mathbf{u} &= \hat{\mathbf{e}}_r \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) + \hat{\mathbf{e}}_\theta \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \\ &\quad + \hat{\mathbf{e}}_z \left( \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right)\end{aligned}$$

*Solution:* We have

$$\nabla \times \mathbf{u} = \left( \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) \times (u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \mathbf{e}_z)$$

$$\begin{aligned}
&= \hat{\mathbf{e}}_r \times \frac{\partial}{\partial r} (u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \mathbf{e}_z) \\
&\quad + \frac{1}{r} \hat{\mathbf{e}}_\theta \times \frac{\partial}{\partial \theta} (u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \mathbf{e}_z) \\
&\quad + \hat{\mathbf{e}}_z \times \frac{\partial}{\partial z} (u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \mathbf{e}_z) \\
&= \hat{\mathbf{e}}_r \times \left( \frac{\partial u_r}{\partial r} \hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial r} \hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial r} \mathbf{e}_z \right) \\
&\quad + \frac{1}{r} \hat{\mathbf{e}}_\theta \times \left( \frac{\partial u_r}{\partial \theta} \hat{\mathbf{e}}_r + u_r \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} + \frac{\partial u_\theta}{\partial \theta} \hat{\mathbf{e}}_\theta + u_\theta \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} + \frac{\partial u_z}{\partial \theta} \mathbf{e}_z \right) \\
&\quad + \hat{\mathbf{e}}_z \times \left( \frac{\partial u_r}{\partial z} \hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial z} \hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial z} \mathbf{e}_z \right) \\
&= \hat{\mathbf{e}}_r \times \left( \frac{\partial u_r}{\partial r} \hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial r} \hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial r} \mathbf{e}_z \right) \\
&\quad + \frac{1}{r} \hat{\mathbf{e}}_\theta \times \left( \frac{\partial u_r}{\partial \theta} \hat{\mathbf{e}}_r + u_r \hat{\mathbf{e}}_\theta + \frac{\partial u_\theta}{\partial \theta} \hat{\mathbf{e}}_\theta - u_\theta \hat{\mathbf{e}}_r + \frac{\partial u_z}{\partial \theta} \mathbf{e}_z \right) \\
&\quad + \hat{\mathbf{e}}_z \times \left( \frac{\partial u_r}{\partial z} \hat{\mathbf{e}}_r + \frac{\partial u_\theta}{\partial z} \hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial z} \mathbf{e}_z \right) \\
&= \frac{\partial u_\theta}{\partial r} (\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta) + \frac{\partial u_z}{\partial r} (\hat{\mathbf{e}}_r \times \mathbf{e}_z) \\
&\quad + \frac{1}{r} \left[ \frac{\partial u_r}{\partial \theta} (\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_r) - u_\theta (\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_r) + \frac{\partial u_z}{\partial \theta} (\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_z) \right] \\
&\quad + \frac{\partial u_r}{\partial z} (\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r) + \frac{\partial u_\theta}{\partial z} (\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_\theta) \\
&= \frac{\partial u_\theta}{\partial r} \hat{\mathbf{e}}_z - \frac{\partial u_z}{\partial r} \hat{\mathbf{e}}_\theta + \frac{1}{r} \left( -\frac{\partial u_r}{\partial \theta} \hat{\mathbf{e}}_z + u_\theta \hat{\mathbf{e}}_z + \frac{\partial u_z}{\partial \theta} \hat{\mathbf{e}}_r \right) \\
&\quad + \frac{\partial u_r}{\partial z} \hat{\mathbf{e}}_\theta - \frac{\partial u_\theta}{\partial z} \hat{\mathbf{e}}_r
\end{aligned}$$

♠ **New Problem 1.4:** Find the divergence of a vector in the cylindrical coordinate system.

*Solution:* We have

$$\begin{aligned}
\nabla \cdot \mathbf{u} &= \left( \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) \cdot (u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \hat{\mathbf{e}}_z) \\
&= \hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_r \frac{\partial u_r}{\partial r} + \hat{\mathbf{e}}_\theta \cdot \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} \frac{u_r}{r} + \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_z \frac{\partial u_z}{\partial z} \\
&= \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = \frac{1}{r} \frac{\partial (r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}
\end{aligned}$$

**1.12** For an arbitrary second-order tensor  $\mathbf{S}$ , show that  $\nabla \cdot \mathbf{S}$  in the cylindrical coordinate system is given by

$$\begin{aligned}\nabla \cdot \mathbf{S} = & \left[ \frac{\partial S_{rr}}{\partial r} + \frac{1}{r} \frac{\partial S_{\theta r}}{\partial \theta} + \frac{\partial S_{zr}}{\partial z} + \frac{1}{r} (S_{rr} - S_{\theta\theta}) \right] \hat{\mathbf{e}}_r \\ & + \left[ \frac{\partial S_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial S_{\theta\theta}}{\partial \theta} + \frac{\partial S_{z\theta}}{\partial z} + \frac{1}{r} (S_{r\theta} + S_{\theta r}) \right] \hat{\mathbf{e}}_\theta \\ & + \left[ \frac{\partial S_{rz}}{\partial r} + \frac{1}{r} \frac{\partial S_{\theta z}}{\partial \theta} + \frac{\partial S_{zz}}{\partial z} + \frac{1}{r} S_{rz} \right] \hat{\mathbf{e}}_z\end{aligned}$$

*Solution:* Using the del operator in the cylindrical coordinate system, the divergence of the tensor  $\mathbf{S}$  is computed as

$$\begin{aligned}\left( \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) \cdot [S_{rr} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_r + S_{r\theta} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_\theta + S_{\theta r} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r + \cdots + S_{zz} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_z] \\ = \frac{\partial S_{rr}}{\partial r} \hat{\mathbf{e}}_r + \frac{\partial S_{r\theta}}{\partial r} \hat{\mathbf{e}}_\theta + \frac{\partial S_{rz}}{\partial r} \hat{\mathbf{e}}_z + \frac{1}{r} \left[ S_{rr} \hat{\mathbf{e}}_\theta \cdot \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} \hat{\mathbf{e}}_r + S_{r\theta} \hat{\mathbf{e}}_\theta \cdot \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} \hat{\mathbf{e}}_\theta \right. \\ \left. + \frac{\partial S_{\theta\theta}}{\partial \theta} \hat{\mathbf{e}}_\theta + S_{\theta\theta} \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} + \frac{\partial S_{\theta r}}{\partial \theta} \hat{\mathbf{e}}_r + S_{\theta r} \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} + S_{zr} \hat{\mathbf{e}}_\theta \cdot \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} \hat{\mathbf{e}}_z + \frac{\partial S_{\theta z}}{\partial \theta} \hat{\mathbf{e}}_z \right] \\ + \frac{\partial S_{zr}}{\partial z} \hat{\mathbf{e}}_r + \frac{\partial S_{z\theta}}{\partial z} \hat{\mathbf{e}}_\theta + \frac{\partial S_{zz}}{\partial z} \hat{\mathbf{e}}_z \\ = \frac{\partial S_{rr}}{\partial r} \hat{\mathbf{e}}_r + \frac{\partial S_{r\theta}}{\partial r} \hat{\mathbf{e}}_\theta + \frac{\partial S_{rz}}{\partial r} \hat{\mathbf{e}}_z + \frac{S_{rr}}{r} \hat{\mathbf{e}}_r + \frac{S_{r\theta}}{r} \hat{\mathbf{e}}_\theta + \frac{1}{r} \frac{\partial S_{\theta\theta}}{\partial \theta} \hat{\mathbf{e}}_\theta - \frac{S_{\theta\theta}}{r} \hat{\mathbf{e}}_r \\ + \frac{1}{r} \frac{\partial S_{\theta r}}{\partial \theta} \hat{\mathbf{e}}_r + \frac{S_{\theta r}}{r} \hat{\mathbf{e}}_\theta + \frac{S_{rz}}{r} \hat{\mathbf{e}}_z + \frac{1}{r} \frac{\partial S_{\theta z}}{\partial \theta} \hat{\mathbf{e}}_z + \frac{\partial S_{zr}}{\partial z} \hat{\mathbf{e}}_r + \frac{\partial S_{z\theta}}{\partial z} \hat{\mathbf{e}}_\theta + \frac{\partial S_{zz}}{\partial z} \hat{\mathbf{e}}_z,\end{aligned}$$

where the following derivatives of the base vectors are accounted for:

$$\frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r, \quad \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta.$$

Collecting the coefficients of  $\hat{\mathbf{e}}_r$ ,  $\hat{\mathbf{e}}_\theta$ , and  $\hat{\mathbf{e}}_z$ , we obtain the required result.

**1.13** For an arbitrary second-order tensor  $\mathbf{S}$  show that  $\nabla \times \mathbf{S}$  in the cylindrical coordinate system is given by

$$\begin{aligned}\nabla \times \mathbf{S} = & \hat{\mathbf{e}}_r \hat{\mathbf{e}}_r \left( \frac{1}{r} \frac{\partial S_{zr}}{\partial \theta} - \frac{\partial S_{\theta r}}{\partial z} - \frac{1}{r} S_{z\theta} \right) + \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta \left( \frac{\partial S_{r\theta}}{\partial z} - \frac{\partial S_{z\theta}}{\partial r} \right) + \\ & \hat{\mathbf{e}}_z \hat{\mathbf{e}}_z \left( \frac{1}{r} S_{\theta z} - \frac{1}{r} \frac{\partial S_{rz}}{\partial \theta} + \frac{\partial S_{\theta z}}{\partial r} \right) + \hat{\mathbf{e}}_r \hat{\mathbf{e}}_\theta \left( \frac{1}{r} \frac{\partial S_{z\theta}}{\partial \theta} - \frac{\partial S_{\theta\theta}}{\partial z} + \frac{1}{r} S_{zr} \right) + \\ & \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r \left( \frac{\partial S_{rr}}{\partial z} - \frac{\partial S_{zr}}{\partial r} \right) + \hat{\mathbf{e}}_r \hat{\mathbf{e}}_z \left( \frac{1}{r} \frac{\partial S_{zz}}{\partial \theta} - \frac{\partial S_{\theta z}}{\partial z} \right) + \\ & \hat{\mathbf{e}}_z \hat{\mathbf{e}}_r \left( \frac{\partial S_{\theta r}}{\partial r} - \frac{1}{r} \frac{\partial S_{rr}}{\partial \theta} + \frac{1}{r} S_{r\theta} + \frac{1}{r} S_{\theta r} \right) + \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_z \left( \frac{\partial S_{rz}}{\partial z} - \frac{\partial S_{zz}}{\partial r} \right) + \\ & \hat{\mathbf{e}}_z \hat{\mathbf{e}}_\theta \left( \frac{\partial S_{\theta\theta}}{\partial r} + \frac{1}{r} S_{\theta\theta} - \frac{1}{r} S_{rr} - \frac{1}{r} \frac{\partial S_{r\theta}}{\partial \theta} \right)\end{aligned}$$